S.T. Yau High School Science Award

Research Report

The Team

Name of team member: Neil Kolekar
School: Stanford Online High School
City, State/Province, Country: Los Altos, California, USA

Name of supervising teacher: Son Nguyen
Job Title: Graduate Student
School/Institution: Massachusetts Institute of Technology
City, State/Province, Country: Cambridge, MA, USA

Title of Research Report

On Support of Temperley-Lieb Immanants of Ribbon Decomposition Matrices

Date

9/15/2026

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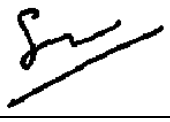
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X 

Name of team member: Neil Kolekar

X 

Name of supervising teacher: Son Nguyen

On Support of Temperley–Lieb Immanants of Ribbon Decomposition Matrices

Neil Kolekar

September 15, 2026

Abstract

A 2005 result of Rhoades and Skandera establishes that Temperley–Lieb immanants, a subset of the dual canonical basis, are Schur-positive when evaluated on Jacobi–Trudi matrices. In 2024, Nguyen and Pylyavskyy established a Littlewood–Richardson rule for Temperley–Lieb immanants of Jacobi–Trudi matrices via a new combinatorial framework of *shuffle tableaux*, yielding a combinatorial proof of their Schur-positivity. This rule was later extended to Temperley–Lieb immanants evaluated on *ribbon decomposition matrices*, a class of matrices which subsume several classical determinantal formulas for skew Schur functions, including the Jacobi–Trudi, Giambelli, and Lascoux–Pragacz identities.

In 2026, Yevtushenko proved that for a restricted class of shapes, the Schur supports of Temperley–Lieb immanants of the corresponding Jacobi–Trudi matrices have a maximal element with respect to the dominance order, and conjectured that the Schur support contains both maximal and minimal elements for all shapes.

By developing a novel local framework on shuffle tableaux, we resolve and generalize Yevtushenko's conjecture by proving that the Schur supports of Temperley–Lieb immanants of all ribbon decomposition matrices contain maximal and minimal elements, each appearing with coefficient one. In particular, we characterize the shuffle tableaux corresponding to the maximal and minimal terms via an inductive procedure for the former and a greedy algorithm for the latter. We conclude with several conjectures that suggest further applications of our local framework.

Keywords: Temperley–Lieb immanants, Schur-positive, dual canonical basis

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1 Introduction

In 1990, Lusztig [Lus90] introduced dual canonical bases. Remarkably, in type A, all elements of these bases satisfy the property of being Schur-positive when evaluated on Jacobi–Trudi matrices [RS06; Hai93]. For example, the determinant is an element of dual canonical bases, and the Jacobi–Trudi identity expresses a given skew Schur function as the determinant of the corresponding Jacobi–Trudi matrix.

The coefficients of the Schur expansion of a given skew Schur function can be determined via the *Littlewood–Richardson rule*, which expresses them combinatorially as the number of Yamanouchi tableaux of given shape and content. Using this rule, and a few other equivalent rules, the Schur supports of skew Schur functions have been studied extensively; see [Aze99; ACM15; GHS17] for examples.

For dual canonical bases in general, there is no known formula for the Schur expansion. However, for a subset of dual canonical basis elements called *Temperley–Lieb immanants*, there does exist a combinatorial description of coefficients in their Schur expansion: in [NP25], Nguyen and Pylyavskyy proved a Littlewood–Richardson rule for Temperley–Lieb immanants of Jacobi–Trudi matrices in terms of *shuffle tableaux*. In [NP26], Nguyen and Pylyavskyy extended their shuffle tableaux rule to a broader class of matrices called *ribbon decomposition matrices*.

A natural research direction that stems from the rule in [NP25] is the study of the Schur support of Temperley–Lieb immanants of generalized ribbon decomposition matrices. Given a symmetric function $f = \sum_\lambda c_\lambda \cdot s_\lambda(x)$, where c_λ are constants indexed by integer partitions, one can consider the set

$$\mathrm{Supp}_s(f) := \{\lambda \mid c_\lambda \neq 0\},$$

which is called the *Schur support* of f. Since the coefficients in the Schur expansion of Temperley–Lieb immanants count various shuffle tableaux, proving that a given partition lies in the Schur support is equivalent to demonstrating the existence of a shuffle tableau that satisfies certain properties.

Recall that the dominance order on integer partitions is defined by

$$\mu \leq_D \lambda \iff \sum_{i=1}^{k} \mu_k \leq \sum_{i=1}^{k} \lambda_k \quad \forall\, k \geq 1.$$

It is known that the Schur support of skew Schur functions has a minimal and maximal element – each of multiplicity 1 – in the dominance order [Aze99]. The property of the Schur support having maximal or minimal elements is often a target for newly defined classes of symmetric functions, as proofs of this property can reveal structural features of the underlying combinatorial objects. In particular, many intermediate results proven in this paper may be of independent interest in the future study of shuffle tableaux. Several classes of polynomials in algebraic combinatorics have a maximal or minimal term in their Schur support with respect to the dominance lattice; classical examples include Stanley symmetric functions [Sta84], double Schubert polynomials [Cas+23], and Lorentzian symmetric polynomials [CQ25].

The study of the Schur support of Temperley–Lieb immanants was initiated by Yevtushenko in [Yev26], in which it was shown that for a special class of shapes, the Schur support contains a minimal element of multiplicity 1.

Theorem 1.1 ([Yev26], Theorem 1.4)**.** *For any strict partitions μ and ν of the same length ℓ such that $\mu_\ell > \nu_1$, let $A_{\mu,\nu}$ be the corresponding Jacobi–Trudi matrix. For any Temperley–Lieb type τ with $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) \neq 0$, there is a maximal partition in the Schur support of $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$.*

Yevtushenko conjectured the following generalization of [Yev26, Theorem 1.4].

Conjecture 1.2 ([Yev26], Conjecture 3.2)**.** *For any partitions μ and ν of the same length ℓ and any Temperley–Lieb type τ with $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) \neq 0$, there is are minimal and maximal partitions in the Schur support of $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$.*

In this paper, we prove that the Schur support of any Temperley–Lieb immanant on any particular ribbon decomposition matrix has both a minimal and maximal element, each of multiplicity 1, proving and generalizing [Yev26, Conjecture 3.2].

Theorem 1.3. *Let λ/μ be a skew shape of length ℓ, R be an infinite ribbon compatible with λ/μ, and τ be a basis element of $\mathrm{TL}_\ell(2)$ with $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$. Then there is* **(i)** *a unique maximal partition and* **(ii)** *a unique minimal partition in the Schur support of $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R})$ with respect to the dominance order, and the coefficient of each corresponding term is* 1.

Remark 1.4. In addition, [Yev26, Conjecture 8.2] conjectures that the minimal tableau $T_{\min}[\tau]$ corresponding to the maximal partition $\lambda_{\max}$ in the Schur support of $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$ is elementwise minimal. We resolve this positively in the generality of ribbon decomposition matrices.

The remainder of this paper is structured as follows. In Section 2, we set up the relevant terminology and background on Temperley–Lieb immanants, ribbon decomposition matrices, and shuffle tableaux used throughout this work. We then establish some preliminary results on Yamanouchi shuffle tableaux in Section 3. Section 4 proves Part (i) of Theorem 1.3 by constructing the minimal tableau though an inductive construction and proving its minimality. Section 5 proves Part (ii) of Theorem 1.3 by constructing the maximal tableau though a greedy algorithm and proving its maximality. Finally, in Section 6, we give an algebraic proof of Theorem 1.3, Part (ii) for the special case of Jacobi–Trudi matrices, and subsequently propose several open problems for potential future applications of our methodology.

Acknowledgements

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2 Background

2.1 Temperley–Lieb immanants

Definition 2.1. The *Temperley–Lieb algebra* $\mathrm{TL}_n(\xi)$ is the $\mathbb{C}[\xi]$-algebra generated by $t_1, \ldots, t_{n-1}$ subject to the relations

$$\begin{cases} t_i^2 = \xi t_i & \text{for } i = 1, \ldots, n-1 \\ t_i t_j t_i = t_i & \text{if } |i-j| = 1 \\ t_i t_j = t_j t_i & \text{if } |i-j| > 1. \end{cases}$$

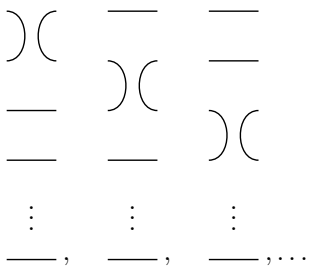
The Temperley–Lieb algebra has a natural basis

$$\{t_w \mid w \text{ is a 321-avoiding permutation}\}.$$

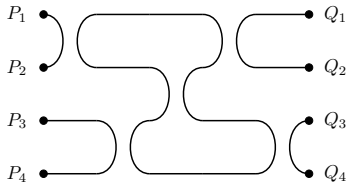
Note this implies that $\dim(TL_n(\xi))$ is the nth Catalan number. Let $f_w(v)$ be the coefficient of $t_w \in \mathrm{TL}_n(2)$ in the basis expansion of $\theta(T_v) = (t_{i_1} - 1)\dots(t_{i_v} - 1)$ for a reduced decomposition $v = s_{i_1}\dots s_{i_v}$. Following [RS05], the *Temperley–Lieb immanant* of a matrix $X = (x_{ij})_{1\le i,j\le n}$ is defined by

$$\mathrm{Imm}_w^{\mathrm{TL}}(X) := \sum_{v\in S_n} f_w(v)\cdot x_{1,v(1)}\dots x_{n,v(n)}.$$

As an alternative to working with the Temperley–Lieb algebra in an algebraic manner, we can work with it combinatorially as follows. First, we require a graphical representation of the Temperley–Lieb algebra $\mathrm{TL}_n(\xi)$. We represent the generators $t_1, t_2, \dots$ graphically as follows.



Concatenating the above diagrams allows us to construct any element of $\mathrm{TL}_n(\xi)$. For a basic example, the identity element of $\mathrm{TL}_n(\xi)$, which corresponds to $t_1^0 t_2^0 \dots$ and has an associated diagram of segments connecting P_i with Q_i for every i, represents the pattern of no intersections. Another example is to consider the following graphical representation of $t_1t_3t_2t_1t_3$, which can be verified by splitting the diagram into the generators of the algebra.



Moreover, there is a natural bijection between Temperley–Lieb basis elements and non-crossing matchings on $2n$ vertices via the graphical representation. Given a 321-avoiding permutation $w \in S_n$, let t_w denote the basis element of $\mathrm{TL}_n(2)$ which corresponds to w under this bijection.

2.2 Ribbon decomposition matrices and shuffle tableaux

Now, we transition to developing a combinatorial understanding of the Temperley–Lieb immanants of ribbon decomposition matrices. We follow the presentation of [NP26]. First, we introduce shuffle shapes and tableaux.

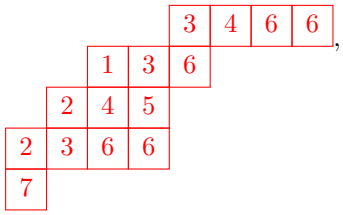
Definition 2.2. Given two skew shapes λ_R/μ_R and λ_B/μ_B, the *shuffle diagram* D of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ is constructed as follows.

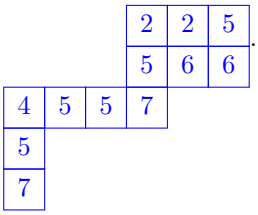
- If λ_R/μ_R has a box in position (i, j), then add into D a box in position $(2i-1, 2j-1)$.
- If λ_B/μ_B has a box in position (i, j), then add into D a box in position $(2i, 2j)$.

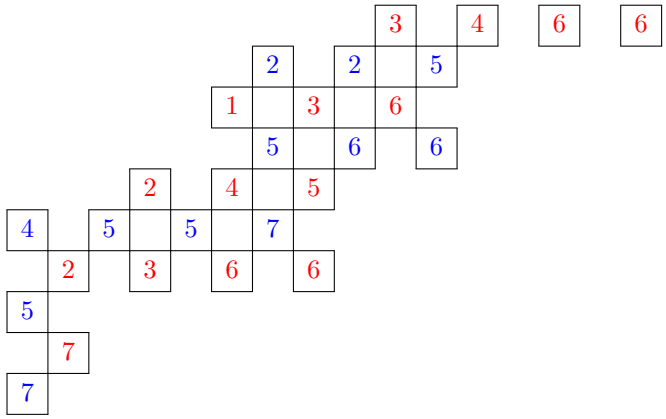
A *shuffle tableau* of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ is a filling of the shuffle diagram of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ such that the entries are weakly increasing along the rows and strictly increasing along the columns. The *weight* of a shuffle tableau T, denoted $\omega(T)$, is $\prod_{i\ge1} x_i^{\alpha_i}$, where α_i is the number of i-entries in T.

Let us put Definition 2.2 into practice in the following example.

Example 2.3. Let T_B be a tableaux of shape $(6,5,3)/(3,1)$ and T_R be a tableaux of shape $(5,4,2)/(2)$, as shown below. The shuffle tableaux T of T_R and T_B is also shown below.

$T_R =$ 

$T_B =$ 

$T =$ 

A *ribbon* is a connected skew shape with no 2×2 block. An *infinite ribbon* is a ribbon with extra $1 \times \infty$ or $\infty \times 1$ blocks at the beginning and the end. Let R be an infinite ribbon, we can tile the plane with infinitely many copies of R. A skew shape λ/μ is *compatible* with R if the intersection of λ/μ with each copy of R is contiguous.

Given a square $(i,j) \in \mathbb{Z}^2$, its *content* is $j - i$. Here, the i-coordinate increases downward while the j-coordinate increases rightward. Every infinite ribbon R has exactly one box of each content. Thus, we can denote $[a,b]_R$, for $a, b \in \mathbb{N}$, to be the section of the ribbon between content a and b inclusive. We denote $(a,b)_R, [a,b)_R, (a,b]_R$ similarly.

Let R be an infinite ribbon and λ/μ be a skew shape compatible with R. From inside out, let $\theta_1, \ldots, \theta_\ell$ be the intersections of λ/μ and copies of R. Each θ_i is an interval $[a_i, b_i)_R$ for some $a_i < b_i$. Hence, we can identify λ/μ with two tuples $\bar{a} = (a_1, a_2, \ldots, a_\ell)$ and $\bar{b} = (b_1, b_2, \ldots, b_\ell)$. We call $\bar{a}$ the ending tuple and $\bar{b}$ the starting tuple of λ/μ.

Definition 2.4. Given an infinite ribbon R and skew shape λ/μ compatible with R. Let $\bar{a}$ and $\bar{b}$ be the ending and starting tuples of λ/μ. The ribbon decomposition matrix of $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with respect to R is

$$A_{\lambda/\mu,R} = \left(s_{[a_j,b_i)_R}\right)_{i,j=1}^{n},$$

with the convention $s_{[a_j,b_i)_R} = 1$ if $a_j = b_i$ and 0 is $a_j > b_i$.

Remark 2.5. The following are some known special cases of ribbon decomposition matrices.

- When R is an infinite $1 \times \infty$ row, every skew shape λ/μ is compatible with R, and $A_{\lambda/\mu,R}$ is the *Jacobi–Trudi matrix*.
- When R is an infinite $\infty \times 1$ column, $A_{\lambda/\mu,R}$ is the *dual Jacobi–Trudi matrix*, or *Näegelsbach–Kostka matrix*.
- When R is an infinite hook, any straight shape λ is compatible with R, and $A_{\lambda,R}$ is the *Giambelli matrix*.
- When R coincides with the outer strip of λ, $A_{\lambda,R}$ is the *Lascoux–Pragacz matrix* in [LP88].

Example 2.6. Figure 1 shows a skew shape with contents of its squares. We have $\bar{a} = (0, -4, -3, 3)$ and $\bar{b} = (3, 5, 9, 6)$.

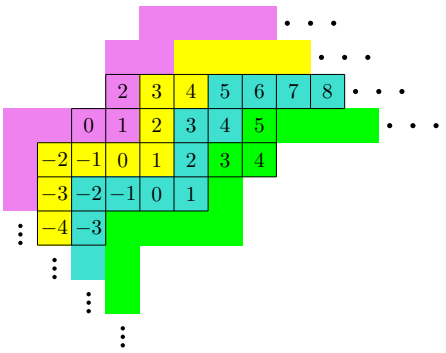
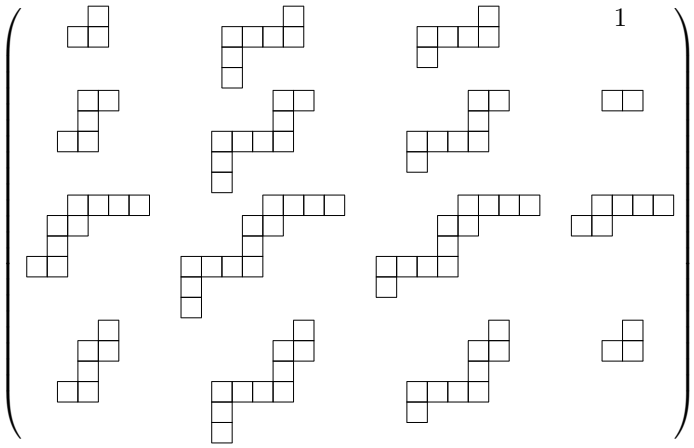


Figure 1: Skew shape and content

The corresponding ribbon decomposition matrix is



Now, we define the notion of the Temperley–Lieb type of a shuffle tableau as follows.

Definition 2.7. In this definition, we explain how to identify the Temperley–Lieb type of a shuffle tableaux T. First, place line segments between squares in accordance to the following rule. If $i \leq j$, then draw



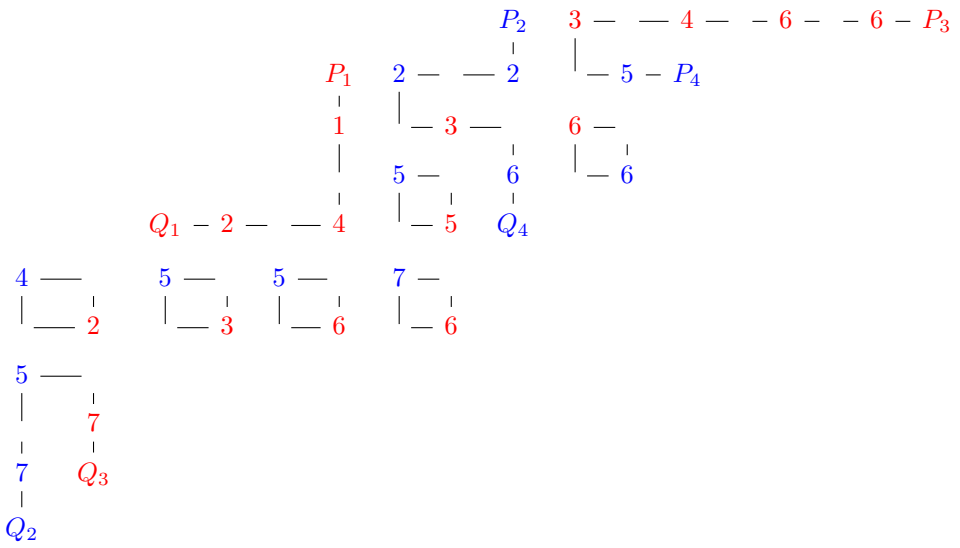
if $i > j$, then draw



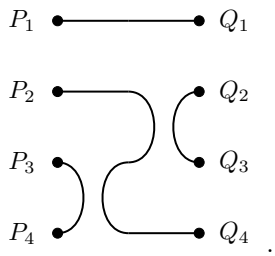
Next, we create the nodes that will serve as vertices of our non-crossing matching. Let $[a_k, b_k)$ be a ribbon in λ/μ, and suppose the box r_{a_k} of this ribbon corresponds to the box (i, j) in $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$. If box $r_{a_{k-1}}$ is below box r_{a_k} in R, then we put Q_k at coordinate $(i + 1, j)$ (below (i, j)). Else, we put Q_k at coordinate $(i, j - 1)$ (left of (i, j)). Similarly, suppose the box $r_{b_{k-1}}$ of this ribbon corresponds to the box (i', j') in $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$. If box r_{b_k} is above box $r_{b_{k-1}}$ in R, then we put P_k at coordinate $(i' - 1, j')$ (above (i', j')). Else, we put P_k at coordinate $(i', j' + 1)$ (right of (i', j')). We call the resulting pattern the *Temperley–Lieb pattern* of the tableau. Observe that the line segments determine a non-crossing matching on the P_i and Q_i. This matching is the *Temperley–Lieb type of* T, which we denote by $\psi(T)$.

Example 2.8. Consider the shuffle tableaux T from Example 2.3. Placing segments in accordance to

Definition 2.7 returns the following.



From here, we can identify the Temperley–Lieb type of T as below.



In [NP26], it is established that Temperley–Lieb immanants are generating functions for shuffle tableaux with prescribed Temperley–Lieb type.

Theorem 2.9 ([NP26], Corollary 3.13)**.** *For any basis element τ of $\mathrm{TL}_n(2)$, we have*

$$\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) = \sum_{\substack{T \text{ of shape } (\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B) \\ \psi(T)=\tau}} \omega(T).$$

It is helpful to define an ordering on the squares of a given shuffle shape, as well as a coordinate system for representing squares.

Definition 2.10 (ordering on squares)**.** Let $(\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$ be a shuffle shape. Given squares $\square, \square' \in (\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$, we write $\square \leq \square'$ if one of the following is true:

- $\square$ lies in a row above $\square'$;
- $\square$ is to the right of $\square'$ in the same row as $\square'$;
- $\square = \square'$.

The coordinate system for the squares of shuffle shapes is defined as follows. We do not rely on a designated square with coordinates $(0,0)$, as we describe the coordinates of squares relative to one another in our proofs.

Definition 2.11 (coordinate system)**.** Let $\square$ be a square of the shuffle shape $(\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$. Then the square immediately below in the same column of $\square$ is given by $(a, b-2)$, and the square immediately to the right of $\square$ is given by $(a+2, b)$.

We also define the notion of orientations of squares.

Definition 2.12. Given a square $\square \in (\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$, its *orientation* of $\square$ is defined to be *horizontal* if there are horizontal line segments connecting $\{\square, \square+(1,0)\}, \{\square+(0,1), \square+(1,1)\}$ and *vertical* if there are vertical line segments connecting $\{\square, \square+(0,1)\}, \{\square+(1,0), \square+(1,1)\}$.

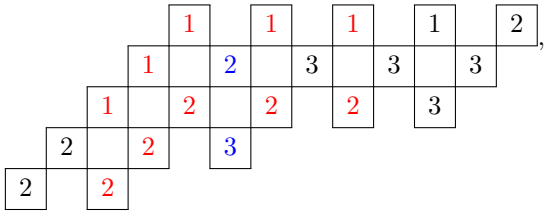
2.3 Temperley–Lieb crystals

First, we introduce reading words of shuffle tableaux and word-Yamanouchi tableaux.

Definition 2.13. Let T be a shuffle tableau. The *reading word* of T is the word given by reading T in accordance to the ordering on the squares of T. A shuffle tableau T is *word-Yamanouchi* if its reading word is a lattice word.

Yamanouchi shuffle tableaux, however, are slightly different from word-Yamanouchi shuffle tableaux. Given a shuffle tableau T, an $(i, i+1)$-*overlap* is a pair of squares (s, t) such that s contains an i, t contains an $i+1$, and s and t are on the same column. If (s, t) is an $(i, i+1)$-overlap pair, we say that s and t are $(i, i+1)$-overlapped.

Example 2.14. In the following tableau T,



there are five pairs $(1, 2)$-overlap (in red), and one pair of $(2, 3)$-overlap (in blue).

Next, we define the i-reading word $w_i(T)$ as follows.

1. Consider the squares consisting of i and $i+1$.
2. Remove all $(i, i+1)$-overlap pairs.
3. Iteratively read the remaining squares from bottom to top, left to right.

Example 2.15. In the tableau in Example 2.14, we have

$$w_1(T) = 2 \mid 2 \mid \mid 2 \mid 1\ 2$$

by reading all non-red 1s and 2s, and

$$w_2(T) = 2\ 2 \mid 2\ 2 \mid 2\ 2\ 2\ 3 \mid 3\ 3\ 3 \mid 2$$

by reading all non-blue 2s and 3s.

Finally, we define the crystal operators on T by acting on $w_i(T)$ in the same fashion as classical crystal operators acting on words.

Definition 2.16 (Crystal operators)**.** The *crystal operators* E_i and F_i act on $w_i(T)$ in the following (standard) way.

- View each i as a closing parenthesis ")" and each $i+1$ as an opening parenthesis "(".
- Match the parentheses in the usual way.
- E_i changes the leftmost unmatched $i+1$ to i, and F_i changes the rightmost unmatched i to $i+1$.

Definition 2.17. A given shuffle tableau T is called *Yamanouchi* if no E_i operator can be applied on T.

Observe that, equivalently, a shuffle tableau T is Yamanouchi if and only if $w_i(T)$ is a lattice word for all i. This can be seen as a strengthened version of the word-Yamanouchi condition, which only requires the entire reading word $w(T)$ to be a lattice word. We will largely work with the word-based definition.

In [NP26], the following Littlewood–Richardson rule for the Schur expansions of Temperley–Lieb immanants of ribbon decomposition matrices is established.

Theorem 2.18 ([NP26], Remark 3.19)**.** *For any skew partition λ/μ, infinite ribbon R compatible with λ/μ, and any partition ν, the coefficient of the Schur function s_ν in $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R})$ is the number of Yamanouchi shuffle tableaux of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, Temperley–Lieb type τ, and content ν.*

Remark 2.19. Observe that when τ is set to the trivial Temperley–Lieb type τ_0 in Theorem 2.18, the result is the classical Littlewood–Richardson rule for the Schur expansion of skew Schur functions. This is because the determinant of the ribbon decomposition matrix $A_{\lambda/\mu,R}$ is the skew Schur function $s_{\lambda/\mu}$, which follows from the fact that ribbon decompositions are special cases of Hamel–Goulden matrices.

Remark 2.20. An efficient way to generate Yamanouchi shuffle tableaux can be seen in [NNW26] via peelable tableaux.

From Theorem 2.18, we establish that a given partition λ is in the Schur support of $\mathrm{Imm}_{\tau}^{\mathrm{TL}}(A_{\mu,\nu})$ if and only if there exists a Yamanouchi shuffle tableaux of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, Temperley–Lieb type τ, and content λ. This gives us a concrete combinatorial target for proving Theorem 1.3. In the following definition, we set up terminology which allows us to refer to contiguous sub-words of reading words.

Definition 2.21. Given a fixed tableau T and a square $\square \in T$, define $\mathsf{w}_i(\square)$ to be the word of i's and $(i+1)$'s that appear when reading the tableau in accordance to the ordering on the squares (see Definition 2.10) until square $\square$ (not including $\square$). Furthermore, define $\mathsf{w}(\square)$ to be the entire word obtained from reading the tableau in accordance to the ordering on the squares until square $\square$ (not including $\square$). For squares $\square_1 < \square_2$, we also define $\mathsf{w}(\square_1, \square_2)$ to be the entire word obtained from reading the tableau in accordance to the ordering on the squares between $\square_1$ and $\square_2$ (not including $\square_1$ nor $\square_2$).

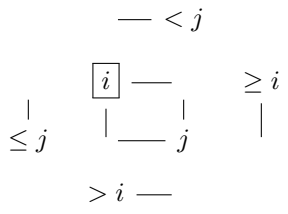
3 Preliminary Results

The main results of this subsection are Lemma 3.2, Proposition 3.3, and Lemma 3.8.

Definition 3.1. In a Temperley–Lieb pattern $\mathbf{S}$, a 2×2 *cycle* is a cycle of the form $\square \to \square + (1,0) \to \square + (1,1) \to \square + (0,1) \to \square$ in $\mathbf{S}$.

Lemma 3.2. *Suppose that $\square$ and $\square + (1,-1)$ are squares of a tableau T of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with entries i and j, respectively. If $i \geq j$, then $\square$ and $\square + (1,-1)$ lie on a 2×2 cycle. In particular, for all $\square \in T$, the entry in $\square$ is strictly less than the entry in $\square + (1,-1)$ when the Temperley–Lieb pattern of T has no 2×2 cycles.*

Proof. We have the following diagram, which proves that $\square$ and $\square + (1,-1)$ lie on a 2×2 cycle.



■

We denote *(UR)* by the pattern below:



For the remainder of this section, we prove properties of shuffle tableaux whose Temperley–Lieb patterns avoid the sub-pattern (UR).

Proposition 3.3. *Let T be a tableau whose Temperley–Lieb pattern does not have sub-pattern* (UR)*. Then T is Yamanouchi if and only if it is word-Yamanouchi.*

Proof. The "only if" direction is clear as the overlapping pairs correspond to cancellations in the parentheses representation of the reading word. Let w denote the reading word of T. We demonstrate that if all prefixes of w are lattice words, then so are all prefixes of w_i.

Let $(\square, \square + (0,-2))$ be a pair of squares with entries i and $i+1$, respectively. It suffices to show that if the entries corresponding to the pair $(\square, \square + (0,-2))$ are removed from w_i, then the resulting word remains lattice. Note that we only need to consider the prefix of w_i ending at $\square + (1,-1)$. Let k denote the largest positive integer such that $\square + (2k-1,-1)$ contains entry $i+1$. We make the following observations.

- By Lemma 3.2 and row monotonicity, the entry in $\square + (2j, 0)$ must be i for $1 \leq j \leq k-1$, as it is weakly bounded below by the entry in $\square$ (which is i) and strictly bounded above by the entry in $\square + (2j+1, -1)$ (which is $i+1$).
- We also observe that all entries to the strict left of $\square$ in the row of $\square$ have entry $\leq i-1$. This is because Lemma 3.2 applied to the pairs $(\square + (-1,-1), \square + (0,-2))$ and $(\square + (-2,0), \square + (-1,-1))$ implies that the entry in $\square + (-2, 0)$ is $\leq i-1$. Thus, these entries do not affect the lattice word condition.
- Similarly, all entries to the strict right of $\square + (0,-2)$ have entry $\geq i+2$. Thus, these entries do not affect the lattice word condition.
- By Lemma 3.2 applied to the pair $(\square + (-1,-1), \square + (0,-2))$, all squares to the strict left of $\square + (1,-1)$ in the row of $\square + (1,-1)$ have entry $\leq i$. Thus, these entries do not affect the lattice word condition.
- Finally, observe that as a consequence of the previous, none of the squares $\square + (2j, 0)$ for $1 \leq j \leq k-1$ are a square in an $(i, i+1)$-overlapping pair.

We conclude from the above observations that w_i remains a lattice word upon removing the pair $(\square, \square + (0,-2))$. The proof is complete. ∎

Definition 3.4. Let $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be a shuffle shape, $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, and w be a word of length $|w| \leq |(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)|$. We say that w is a *partial word of* $\mathbf{S}$ if the partial filling of $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ up to the $|w|$th square in the reading order satisfies the tableau conditions (row and column monotonicity), Temperley–Lieb pattern condition (diagonal inequalities), and the Yamanouchi condition. The partial filling associated with a partial word w is called a *partial tableau* of $\mathbf{S}$.

Definition 3.5. Let $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be a shuffle shape, $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, and w be a partial word of $\mathbf{S}$. Given a square $\square \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, we say that $\square$ is a *bottleneck of* $\mathbf{S}$ if there exists a positive integer x such that

- the entries in $\square + (0,2)$ and $\square + (1,1)$ in the partial filling of $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ determined by w are $x+1$ and x, respectively, and
- the word $\mathsf{w}(\square)$ is $(x, x+1)$-balanced.

Furthermore, we say that $\square$ is a *bottlecap* of $(\mathbf{S}, (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B), w)$ if the orientations of $\square + (1,1)$ and $\square + (2,0)$ are horizontal and vertical, respectively.

Lemma 3.6. *Let* $\mathbf{S}$ *be a Temperley–Lieb pattern that avoids the sub-pattern* (UR). *Suppose that* $\square$ *is a bottleneck of* $(\mathbf{S}, (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B), w)$. *Then, either* $\square + (2,0)$ *is a bottleneck of* $\mathbf{S}$, $\square + (1,1)$ *is a bottleneck of* $\mathbf{S}$, *or* $\square$ *is a bottlecap of* $\mathbf{S}$.

Proof. Let x denote the entry in $\square + (1,1)$. We will show that:

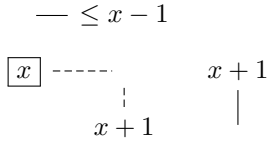
(i) $\square + (2,0)$ contains x if and only if $\square + (3,-1)$ contains $x+1$;

(ii) $\square + (1,1)$ contains x if and only if $\square + (2,0)$ contains $x+1$.

Proof of Part (i). For the "if" direction, suppose that $\square + (3,-1)$ contains $x+1$. By Lemma 3.2, the entry in $\square + (2,0)$ is $\leq x$. Hence, the entry in $\square + (2,0)$ must be x as it is $\geq x$ from row monotonicity, proving the "only if" direction.

For the "only if" direction, suppose that $\square + (2,0)$ contains x. By row monotonicity, the entry in $\square + (3,-1)$ is $\geq x+1$. Assume for contradiction that the entry in $\square + (3,-1)$ is $\geq x+2$. Then, $\mathsf{w}(\square + (1,-1), \square + (2,0))$ has more x's than $(x+1)$'s, which implies that $\mathsf{w}(\square + (4,0))$ has fewer x's than $(x+1)$'s, a contradiction. Thus, the "if" direction holds.

Proof of Part (ii). For the "if" direction, suppose that $\square + (2,0)$ contains $x+1$. By Lemma 3.2, the entry in $\square + (1,1)$ is $\leq x$. Assume for contradiction that the entry in $\square + (1,1)$ is $\leq x-1$. Then, we have the following diagram, which contradicts the avoidance of (UR).



Thus, the "if" direction holds.

For the "only if" direction, suppose that $\square + (1,1)$ contains x. By Lemma 3.2, the entry in $\square + (2,0)$ is $\geq x+1$. Assume for contradiction that the entry in $\square + (2,0)$ is $\geq x+2$. Then, $\mathsf{w}(\square + (1,-1), \square + (1,1))$ has more x's than $(x+1)$'s, which implies that $\mathsf{w}(\square + (3,1))$ has fewer x's than $(x+1)$'s, a contradiction. Hence, the "only if" direction holds. ■

From Lemma 3.6, we can achieve a stronger result, which asserts that the configurations appearing in its premise cannot exist.

Definition 3.7. Let T be a tableau whose Temperley–Lieb pattern has no 2×2 cycles. Note that given a square $\square \in T$, the squares $\square + (1,1)$ and $\square + (2,0)$ must have distinct entries. Thus, for a square $\square \in T$ containing x, we may define $\mathsf{h}(\square)$ to be the unique square in T such that

(i) there exists a unique sequence $(\square_1, \ldots, \square_k)$ of squares containing x with $\square_1 = \square$ and $\square_k = \mathsf{h}(\square)$ satisfying $\square_{j+1} \in \{\square_j + (1,1), \square_j + (2,0)\}$ for all $1 \leq j \leq k-1$, and

(ii) neither $\mathsf{h}(\square) + (1,1)$ nor $\mathsf{h}(\square) + (2,0)$ contain x.

We use $k(\square)$ to formally denote the length of the sequence $(\square_1, \ldots, \square_k)$.

Lemma 3.8. *Suppose that T is a tableau of Temperley–Lieb type τ and $\square$ is a square of T containing x such that $\square + (1,-1)$ contains $x+1$ and $\mathsf{w}(\square + (-1,-1))$ is $(x, x+1)$-balanced. Then, the Temperley–Lieb pattern of T is not $\mathbf{S}_{\max}(\tau)$.*

Proof. Assume for contradiction that the Temperley–Lieb pattern of T is $\mathbf{S}_{\max}(\tau)$. First, we claim that $\mathsf{h}(\square) + (1,-1) = \mathsf{h}(\square + (1,-1))$. We proceed by induction on $k(\square) \geq 1$. For the base case of $k(\square) = 1$, observe that by Lemma 3.6, neither $\square + (3,-1)$ nor $\square + (2,0)$ have entry $x+1$, so $\mathsf{h}(\square + (1,-1)) = \square + (1,-1)$. Now, suppose that $k(\square) \geq 2$. We have two cases.

- If $\square + (1,1)$ has entry x, then $\square + (2,0)$ has entry $x+1$ by Lemma 3.6. Since $\mathsf{w}((\square + (1,1)) + (-1,-1))$ is $(x, x+1)$-balanced and $k(\square + (1,1)) = k(\square) - 1$, we have

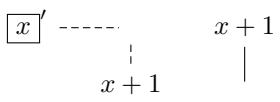
$$h(\square) = \mathsf{h}(\square + (1,1)) = h(\square + (2,0)) = h(\square + (1,-1)),$$

  completing the induction for this case.

- The proof for the case in which $\square + (2,0)$ has entry x is analogous to the previous case.

Therefore, $\mathsf{h}(\square) + (1,-1) = \mathsf{h}(\square + (1,-1))$.

Now, let $\square' := \mathsf{h}(\square)$; note that $\square' + (2,0)$ has entry $\geq x+1$ and $\square' + (1,1)$ has entry $\leq x-1$ by condition (ii) of Definition 3.7. Then, we have the following diagram, which contradicts the avoidance of (UR), completing the proof.



■

4 Minimal Yamanouchi Shuffle Tableau

4.1 Proof outline

We first briefly describe Yevtushenko's approach in [Yev26] towards proving the existence of a maximal partition in the Schur supports of Temperley–Lieb immanants of Jacobi–Trudi matrices corresponding to a special class of shuffle shapes. Note that by Theorem 2.18, this is equivalent to constructing a *minimal* Yamanouchi shuffle tableau of prescribed shape, Temperley–Lieb type, and content. An outline of Yevtushenko's strategy is the following.

1. Define the *canonical minimal strings* $\mathbf{S}_{\min}(\tau)$ via an explicit geometric construction.

2. Prove that if a tableau T has Temperley–Lieb pattern different from $\mathbf{S}_{\min}(\tau)$, then there exists a tableau T' whose content is strictly greater than that of T. This is done by showing that the construction of canonical minimal strings is maximally pushed down.

3. Construct a Yamanouchi tableau $T_{\min}[\tau]$ with Temperley–Lieb pattern $\mathbf{S}_{\min}(\tau)$ which is minimal across all tableaux with Temperley–Lieb pattern $\mathbf{S}_{\min}(\tau)$.

Now, we discuss the construction of $T_{\min}[\tau]$, given the canonical minimal strings $\mathbf{S}_{\min}(\tau)$.

Definition 4.1 ([Yev26], Definition 6.4)**.** Given a shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and a Temperley–Lieb pattern $\mathbf{S}$ on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, define the *ascension graph* $G_{\lambda/\mu}(\mathbf{S})$ by the directed graph on the squares of $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ where we place a directed edge $\square \to \square'$ if and only if either

- $\square' = \square + (0, -2)$ and the orientation of $\square'$ is vertical, or
- $\square' = \square + (-1, -1)$ and the orientation of $\square'$ is horizontal.

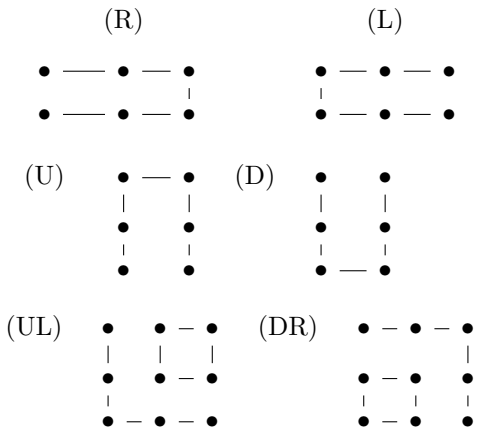
We define $\mathsf{f}_{\mathbf{S}}(\square)$ to be one more than the depth of $\square$ in $G_{\lambda/\mu}(\mathbf{S})$. The tableau $T_{\min}(\mathbf{S})$ is given by the filling of $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ in which the entry of $\square$ is $\mathsf{f}_{\mathbf{S}_{\min}(\tau)}(\square)$.

The approach in [Yev26] does not directly extend to the Temperley–Lieb immanants of Jacobi–Trudi matrices (and more generally, ribbon decomposition matrices) corresponding to *all* skew shapes. This is because for general shuffle shapes, the proof of the well definedness and minimality of the constructed minimal tableau relies heavily on the geometric nature of the endpoints [Yev26, Remark 4.3]. One may observe that much of the difficulty in finding a direct, geometric construction of canonical strings for general shuffle shapes comes from the fact that it is difficult to characterize shuffle shapes $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ for which a given Temperley–Lieb immanant $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R})$ is nonzero.

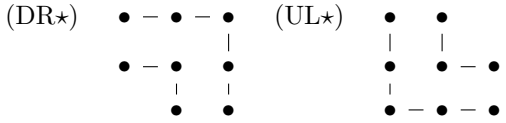
We take the following approach. In Proposition 4.5, we describe the minimal canonical strings as the unique Temperley–Lieb pattern satisfying a set of pattern avoidances, which we construct by repeatedly locally perturbing an arbitrary Temperley–Lieb pattern of type τ, and prove that at most one such pattern exists. Next, we show that the construction $T_{\min}[\tau]$ given by the ascension graph construction for the minimal canonical strings is a Yamanouchi tableau (Proposition 4.6). Our proof of minimality consists of two parts. The first part (Lemma 4.9) shows that $T_{\min}[\tau]$ is the elementwise minimal tableau with Temperley–Lieb pattern $\mathbf{S}_{\max}(\tau)$. In the second part (Proposition 4.10), we show that, given a tableau T whose Temperley–Lieb pattern contains a forbidden sub-pattern, one can construct an elementwise smaller tableau; our construction is based on local changes that are made near the first occurrence of a forbidden sub-pattern.

4.2 Canonical minimal strings

Definition 4.2. Given a shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and Temperley–Lieb type τ, we define the *canonical minimal strings* $\mathbf{S}_{\min}(\tau)$ to be the unique Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with type τ which avoids the six patterns below.

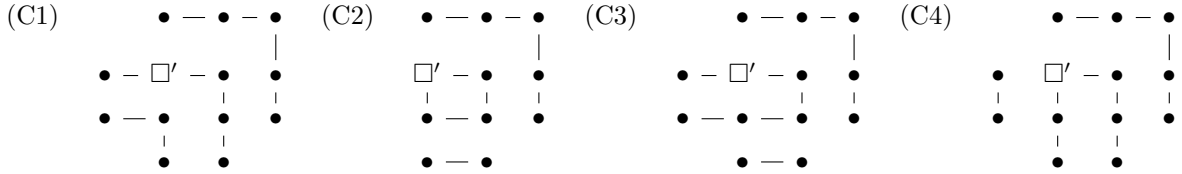


We define two additional patterns (DR$\star$) and (UL$\star$) below.



Lemma 4.3. *Let* $\mathbf{S}$ *be a Temperley–Lieb pattern on* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *which avoids the patterns* (R) *and* (U)*. If* $\mathbf{S}$ *has a* (DR$\star$) *sub-pattern, then it has a* (DR) *sub-pattern, and if* $\mathbf{S}$ *has a* (UL$\star$) *sub-pattern, then it has a* (UL) *sub-pattern.*

Proof. Suppose that $\mathbf{S}$ has a (DR$\star$) sub-pattern; we will show that $\mathbf{S}$ has a (DR) sub-pattern. Let $\square$ be a square in $\mathbf{S}$ such that $\square$ lies in the first row of a (DR$\star$) sub-pattern. Let j be the greatest nonnegative integer such that $\square + (-j, -j)$ lies in the first row of a (DR$\star$) sub-pattern, and define $\square' := \square + (-j-1, -j-1)$. Note that $\square' + (1, -1) \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, or else the orientation of $\square + (1, -1)$ in $\mathbf{S}$ is horizontal. Thus, we have one of the following four cases.

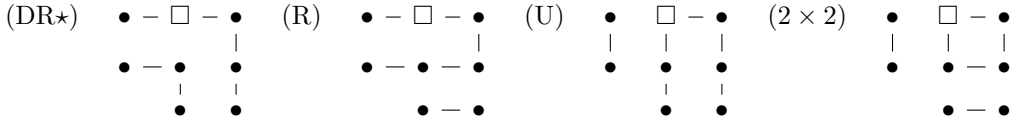


Observe that (C1) contradicts the maximality of j (as $\square' = \square + (-j-1, -j-1)$ would also in the top row of a (DR$\star$) sub-pattern), (C3) contradicts the avoidance of (R), and (C4) contradicts the avoidance of (U). Thus, only the case (C2) remains, so $\mathbf{S}$ has a (DR) sub-pattern, as desired.

The proof that $\mathbf{S}$ has a (UL$\star$) sub-pattern given that it contains a (UL) sub-pattern is analogous. ∎

Lemma 4.4. *Suppose that the Temperley–Lieb pattern* $\mathbf{S}$ *on* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *contains no* (R)*,* (U)*, or* (DR) *sub-patterns. For every square* $\square \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *such that* $\square$ *and* $\square + (1, -1)$ *does not lie on a common* 2×2 *cycle, there is an endpoint* $R \in \{P_i, Q_i : 1 \le i \le \ell\}$ *such that* $\square + (1, 0)$ *and* R *are connected by an up-right lattice path in* $\mathbf{S}$*. In particular, all cycles in* $\mathbf{S}$ *are* 2×2 *cycles.*

Proof. First, we claim that if the pairs $(\square, \square + (1, 0))$ and $(\square + (1, 0), \square + (1, -1))$ are connected in $\mathbf{S}$, then $\square$ lies on a 2×2 cycle. Note that one of the three configurations below appears in $\mathbf{S}$.

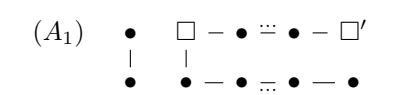


The left three patterns above fail to satisfy the pattern avoidance condition on $\mathbf{S}$; thus, $\square$ and $\square + (1, -1)$ lie on a common 2×2 cycle.

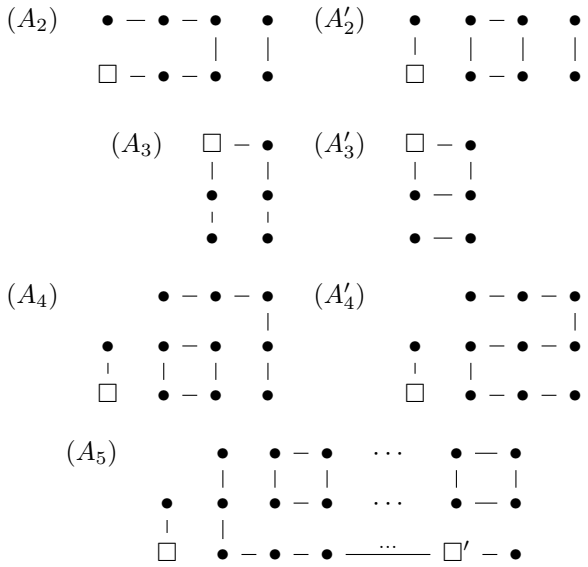
Now, let $\square \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be a square such that $\square$ and $\square + (1, -1)$ do not lie on a common 2×2 cycle. It suffices to show that the point $p := \square + (1, 0)$ satisfies the property that one of the pairs $(p, p + (0, 1))$ and $(p, p + (1, 0))$ is connected in $\mathbf{S}$. If the orientation of $\square$ is vertical, then $(p, p + (0, 1))$ is connected in $\mathbf{S}$. Otherwise, the orientation of $\square$ is horizontal, so the the orientation of $\square + (1, -1)$ is also horizontal. In this case, the pair $(p, p + (1, 0))$ is connected in $\mathbf{S}$, completing the proof. ∎

Proposition 4.5. *Suppose that* τ *is a Temperley–Lieb type on* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *such that* $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$*. Then, there is a unique Temperley–Lieb pattern* $\mathbf{S}$ *on the shape* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *with type* τ *such that* $\mathbf{S}$ *avoids the patterns from Definition 4.2. That is, Definition 4.2 is well defined.*

Proof. Since $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$, there exists a Temperley–Lieb pattern $\mathbf{S}_0$ on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$. Consider the following sub-patterns (A_i) and their corresponding replacement sub-patterns (A_i'). Here, we assume that no forbidden pattern appears prior to $\square$.

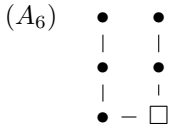


In the above diagram for (A_1), we define $\square' = \square + (2j, 0)$ to satisfy the property that one of $\square'$ and $\square' + (0, -2)$ is of vertical orientation. We then define (A_1') to be $\mathbf{S}$ with the orientations of $\square + (2k + 1, -1)$ toggled for all $k \in [0, j - 1]$; these toggles clearly preserve Temperley–Lieb type.

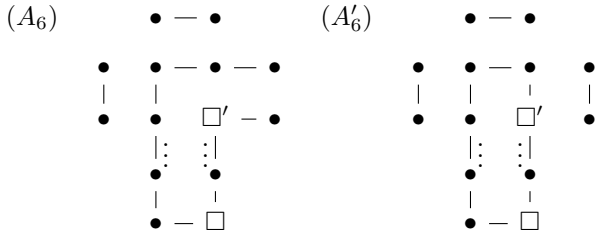


In the above diagram for (A_5), we define $\square' = \square + (2j, 0)$ to satisfy the property that $\square' + (0, 2)$ is not the bottom-left corner of a (DR) sub-pattern. We then define (A_5') to be $\mathbf{S}$ with the orientations of $\square + (2k, 0)$ toggled for all $k \in [1, j]$; these toggles clearly preserve Temperley–Lieb type.

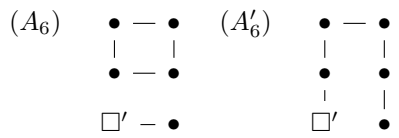
The last sub-pattern, (A_6'), is difficult to display geometrically as above, so we instead describe it directly as follows. First, we define (A_6) to be the pattern below.



Let j be the smallest positive integer for which $\square' := \square + (0, 2j)$ has horizontal orientation. By the avoidance of (UR), the orientation of $\square + (-2, 2k)$ is vertical for all k. We define (A_6') by toggling the orientation of $\square'$. We claim that this toggle preserves Temperley–Lieb type. To show this, we split into cases on the orientation of $\square' + (-1, 1)$. If the orientation of $\square' + (-1, 1)$ is horizontal, then we have the following diagram; we see that the Temperley–Lieb type is preserved as $\square'$ and $\square' + (0, 1)$ are connected by the path $\square' \to \square \to \square + (-1, 0) \to \square' + (-1, 1) \to \square' + (0, 1)$.



Now, suppose the orientation of $\square' + (-1, 1)$ is vertical. By the avoidance of (DR$\star$) (and thus the avoidance of (DR) by Lemma 4.3), $\square' + (0, 2)$ and $\square' + (1, 1)$ lie on a common 2×2 cycle. In this case, we have the following diagram, in which it is clear that the Temperley–Lieb type is preserved.

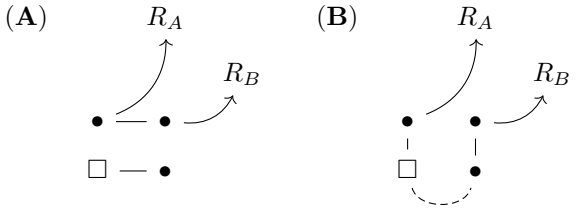


Given a Temperley–Lieb pattern $\mathbf{S}$, denote by $I_A(\mathbf{S})$ the set of squares $\square$ such that one of the six patterns among $(A_1), \ldots, (A_6)$ appears in $\mathbf{S}$. Now, for $\mathbf{S}$ that do not avoid one of the patterns in Definition 4.2, consider the local operation $\mathsf{A} : \mathbf{S} \mapsto \mathsf{A}(\mathbf{S})$ in which the pattern (A_i) associated with $\min(I_A(\mathbf{S}))$ is replaced by the pattern (A_i'). Note that

- A preserves Temperley–Lieb type;
- $\min(I_A(\mathsf{A}(\mathbf{S}))) > \min(I_A(\mathbf{S}))$;
- $\mathsf{A}(\mathbf{S})$ is defined if and only if $\mathbf{S}$ contains at least one of the six forbidden sub-patterns.

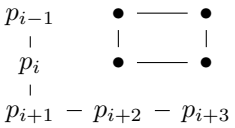
Let N denote the greatest positive integer for which $\mathsf{A}^N(\mathbf{S})$ is defined; note that such N exists due to the increasing nature of $\min(I_A(\mathbf{S}))$ under A. By the maximality of N, the pattern $\mathsf{A}^N(\mathbf{S})$ is a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ of Temperley–Lieb type τ that avoids the patterns in Definition 4.2, so the construction is complete.

Next, we claim that there is at most one Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ of Temperley–Lieb type τ which avoids the patterns in Definition 4.2. Assume for contradiction that there exist Temperley–Lieb patterns $\mathbf{A} \neq \mathbf{B}$ on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ of the same Temperley–Lieb type that both avoid the patterns in Definition 4.2. Let $\square$ be the minimal square in $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ such that orientation for $\square$ in $\mathbf{A}$ and the orientation for $\square$ in $\mathbf{B}$ differ; assume without loss of generality that these are horizontal and vertical, respectively. By Lemma 4.4, there exist endpoints $R_A, R_B \in \{P_i, Q_i : 1 \leq i \leq \ell\}$ such that $\square + (1,0)$ is connected to R_A by an up-right lattice path and $\square + (3,0)$ is connected to R_B by an up-right lattice path (note that these statements hold in both $\mathbf{A}$ and $\mathbf{B}$ by the minimality of $\square$). Since $\{R_A, R_B\}$ is connected in $\mathbf{A}$, it must be connected in $\mathbf{B}$ as well since $\mathbf{A}$ and $\mathbf{B}$ have the same Temperley–Lieb type. In particular, the pair $(\square, \square + (1,0))$ is connected in $\mathbf{B}$ by a path $\square = p_1 \to \cdots \to p_k = \square + (1,0)$.



Note that there exists an index $1 \leq i \leq k-2$ such that $p_{i+1} = p_i + (0,-1)$ and $p_{i+2} = p_{i+1} + (1,0)$. We perform casework on the orientations of p_i and p_{i+2}.

- Case 1: Suppose the orientations of p_i and p_{i+2} are horizontal and vertical, respectively. Then p_i and p_{i+2} lie on a common 2×2 cycle, a contradiction.
- Case 2: Suppose the orientations of p_i and p_{i+2} are both horizontal. Then, we have a contradiction to the avoidance of (L).
- Case 3: Suppose the orientations of p_i and p_{i+2} are both vertical. Then, we have a contradiction to the avoidance of (D).
- Case 4: Suppose the orientations of p_i and p_{i+2} are vertical and horizontal, respectively. Since $p_i + (1,1)$ and $p_{i+2} + (1,1)$ do not lie on the path $p_1 \to \cdots \to p_k$, neither point is connected to an endpoint in $\{P_i, Q_i : 1 \leq i \leq \ell\}$. Thus, by Lemma 4.4, $p_i + (1,1)$ and $p_{i+2} + (1,1)$ lie on a common 2×2 cycle. This yields the following configuration, which contradicts the avoidance of (UL).



Hence, we have a contradiction in all cases above, so the desired uniqueness holds. ■

4.3 Construction

We will construct the maximal tableau $T_{\min}[\tau]$ from the following algorithm, which is an inductive rendition of the construction from ascension graphs.

Algorithm 1. *Given a shape $(\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$ and Temperley–Lieb pattern $\mathbf{S}$, we define the filling $T_{\min}(\mathbf{S})$ of shape $(\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$ by the following process.*

- *Go through the cells starting with the second row from right to left.*
- *Assign entry 0 to all squares $\square \notin (\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$.*
- *For each square $\square$ in the second row or below, fill $\square$ with $x+1$, where x is the entry in:*

  (i) $\square+(0,2)$, if $\square$ has vertical orientation in $\mathbf{S}$;

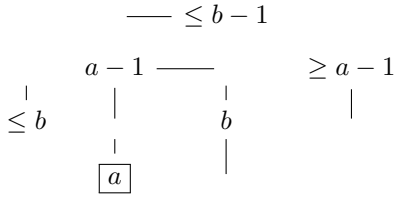
  (ii) $\square+(1,1)$, if $\square$ has horizontal orientation in $\mathbf{S}$.

We denote the partial filling $T_{\min}(\mathbf{S})$ up to and including $\square \in (\lambda_R/\mu_R)\circledast(\lambda_B/\mu_B)$ by $T_{\min}(\mathbf{S},\square)$. We denote the filling $T_{\min}(\mathbf{S}_{\min}(\tau))$ by $T_{\min}[\tau]$.

Proposition 4.6. *Let $\mathbf{S}$ be a Temperley–Lieb pattern for which partial pattern given by the line segments for the squares less than or equal to $\square$ avoids the patterns* (U), (R), *and* (DR). *Then the filling $T_{\min}(\mathbf{S},\square)$ is a Yamanouchi partial tableau of $\mathbf{S}$.*

Proof. Suppose we are in the stage of filling in square $\square$.

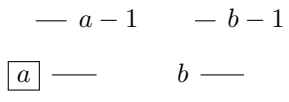
First, we check that in the case that the pair of line segments for $(\square,\square+(1,1))$ are vertical, the entry in $\square$ is at most the entry in $\square+(1,1)$. Let $\square$ have entry a and $\square+(1,1)$ have entry b. Assume for contradiction that $a>b$. Then, we have the following diagram, which contradicts the avoidance of (U).



Thus, the entry in $\square$ is at most the entry in $\square+(1,1)$.

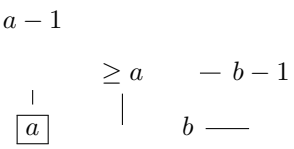
Now we check that row monotonicity holds. Suppose that $\square$ is not the rightmost entry in its row; we will show that the entry in $\square$ is at most the entry in $\square+(2,0)$. Let a be the entry in $\square$ and b be the entry in $\square+(2,0)$, and assume for contradiction that $a>b$. We have the following diagrams for each of the $2^2=4$ cases of line segments for the pairs $(\square,\square+(1,1))$ and $(\square+(2,0),\square+(3,1))$.

1.



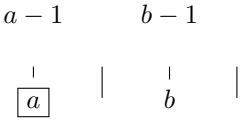
We have a contradiction as $a-1\leq b-1$ by row monotonicity on $\square+(1,1)$ and $\square+(3,1)$.

2.



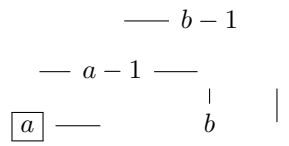
We have a contradiction as $a\leq b-1$ by row monotonicity on $\square+(1,1)$ and $\square+(3,1)$.

3.



We have a contradiction as $a-1\leq b-1$ from row monotonicity on $\square+(0,2)$ and $\square+(2,2)$.

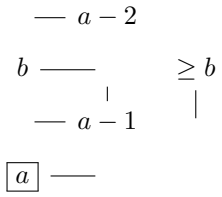
4.



By casework on the orientation of $\square + (1, -1)$, we have a contradiction from the avoidance of (R) and (DR$\star$) (by Lemma 4.3).

Thus, row monotonicity holds.

Finally, we check that column monotonicity holds. Suppose that $\square$ is in the third row or below. Let $\square$ have entry a and $\square + (0, 2)$ have entry b. Assume for contradiction that $a \leq b$; then, we must have the following configuration.



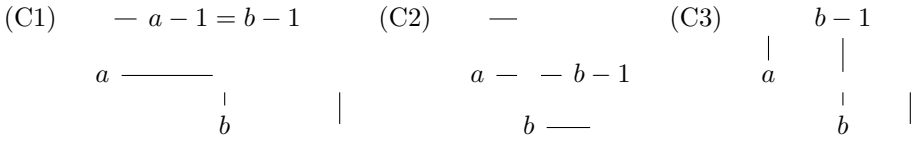
By considering the orientation of $\square + (0, 2)$, we have $b = (a - 2) + 1 < a$, a contradiction. So column monotonicity holds.

Therefore, Algorithm 1 is well defined. To complete the proof, we briefly explain why $T_{\min}[\tau]$ is Yamanouchi. ■

Lemma 4.7. *Let* $\mathbf{S}$ *be a Temperley–Lieb pattern for which partial pattern given by the line segments for the squares less than or equal to* $\square$ *avoids the patterns* (U), (R), *and* (DR). *If* $\square_0 \leq \square$ *and* $\square_0 + (-1, 1) \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, *then the entry of* $\square_0$ *in* $T_{\min}(\mathbf{S}, \square)$ *is greater than or equal to the entry of* $\square_0 + (-1, 1)$ *in* $T_{\min}(\mathbf{S}, \square)$.

Proof. Let a and b denote the entries of $\square_0 + (-1, 1)$ and $\square_0$ in $T_{\min}(\mathbf{S}, \square)$. We split into cases on the orientations of $\square_0$ and $\square_0 + (-1, 1)$.

- Case 1: Suppose that the orientations of $\square_0$ and $\square_0 + (-1, 1)$ are vertical and horizontal, respectively. Then a and b are both 1 more than the entry in $\square_0 + (0, 2)$, so $a = b$.
- Case 2: Suppose that the orientation of $\square_0$ is horizontal. By Proposition 4.6, row monotonicity applies to $\square_0 + (-1, 1)$ and $\square_0 + (1, 1)$, from which $a \leq b - 1$.
- Case 3: Suppose that the orientation of $\square_0$ and $\square_0 + (-1, 1)$ are vertical and vertical, respectively. By Proposition 4.6, the diagonal inequality applies to $\square_0 + (-1, 1)$ and $\square_0 + (0, 2)$, from which $a \leq b - 1$.



Since $a \leq b$ holds in all cases above, the lemma holds. ■

A corollary of Lemma 4.7 is the following, which will prove to be a powerful tool in proving the minimality of $\mathbf{S}_{\min}(\tau)$.

Corollary 4.8. *Let* $\mathbf{S}$ *be a Temperley–Lieb pattern for which partial pattern given by the line segments for the squares less than or equal to* $\square$ *avoids the patterns* (U), (R), *and* (DR). *Let* $\mathbf{S}'$ *be the Temperley–Lieb pattern obtained from toggling the orientation of a square* $\square' \leq \square$ *whose orientation is horizontal. Then,* $T_{\min}(\mathbf{S}')$ *is elementwise smaller than* $T_{\min}(\mathbf{S})$.

Proof. It suffices to show that $\mathsf{f}_{\mathbf{S}'}(\square') \leq \mathsf{f}_{\mathbf{S}}(\square')$. By Lemma 4.7, we have

$$\mathsf{f}_{\mathbf{S}'}(\square') = \mathsf{f}_{\mathbf{S}'}(\square' + (0, 2)) + 1 \leq \mathsf{f}_{\mathbf{S}'}(\square' + (1, 1)) + 1 = \mathsf{f}_{\mathbf{S}}(\square'),$$

as desired. ■

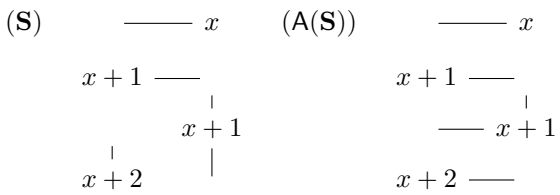
Lemma 4.9. *The tableau $T_{\min}[\tau]$ is the unique minimal Yamanouchi tableau with Temperley–Lieb pattern $\mathbf{S}_{\min}(\tau)$.*

Proof. Let T be a tableau of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with Temperley–Lieb pattern $\mathbf{S}_{\min}(\tau)$. Note that the entry of $\square$ in T is at least 1 more than the depth of $\square$ in $G_{\lambda/\mu}(\mathbf{S})$. Since the entry of $\square$ in $T_{\min}[\tau]$ is exactly 1 more than the depth of $\square$ in $G_{\lambda/\mu}(\mathbf{S})$, the tableau T is elementwise greater than $T_{\min}[\tau]$, completing the proof. ∎

Proposition 4.10. *Let T be a tableau of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and Temperley–Lieb type τ with Temperley–Lieb pattern different from $\mathbf{S}_{\min}(\tau)$. Then, T is elementwise greater than $T_{\min}[\tau]$.*

Proof. Let $\mathbf{S}$ be the Temperley–Lieb pattern of T. It suffices to show that for all squares $\square \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, $\mathsf{f}_{\mathbf{S}}(\square) \geq \mathsf{f}_{\mathsf{A}(\mathbf{S})}(\square)$. We perform casework on the pattern (A_i) associated with $\square_{\min} := \min(I_A(\mathbf{S}))$. Note that for $i \in \{1, 2, 5, 6\}$, the pattern $\mathsf{A}(\mathbf{S})$ can be obtained from toggling squares in $\mathbf{S}$ of horizontal orientation, in which no toggle is performed with a forbidden pattern appearing prior to the toggled square. Thus, by Corollary 4.8, we have $f_{\mathbf{S}}(\square) \geq \mathsf{f}_{\mathsf{A}(\mathbf{S})}(\square)$ for all squares $\square$ when $i \in \{1, 2, 5, 6\}$.

Now, we separately resolve $i = 3$ and $i = 4$. In both cases, A operates in the following form:



Thus, the content of T remains invariant under A for the cases of $i = 3$ and $i = 4$, so we conclude that $f_{\mathbf{S}}(\square) \geq \mathsf{f}_{\mathsf{A}(\mathbf{S})}(\square)$ for all squares $\square$. ∎

4.4 Proof of Theorem 1.3, Part (i)

We now prove the first half of Theorem 1.3.

Proof of Theorem 1.3, Part (i). By Lemma 4.9, $T_{\min}[\tau]$ is the unique minimal Yamanouchi tableau among those with Temperley–Lieb pattern $\mathbf{S}_{\min}(\tau)$. Since all Yamanouchi tableaux with Temperley–Lieb type τ and Temperley–Lieb pattern different from $\mathbf{S}_{\max}(\tau)$ have content strictly less than $\lambda(T_{\max}[\tau])$ by repeated application of Proposition 4.10, we conclude that $T_{\min}[\tau]$ is the unique minimal tableau of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and Temperley–Lieb type τ. ∎

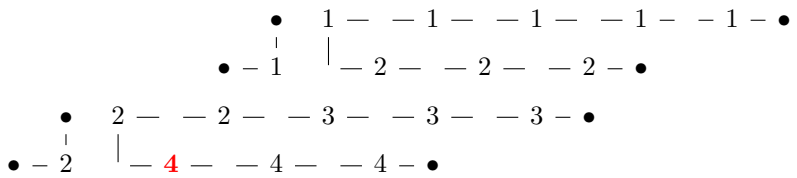
5 Maximal Yamanouchi Shuffle Tableau

5.1 Proof outline

We now provide an outline of the proof of Part (ii) of Theorem 1.3. We describe the maximal tableau by the result of a greedy algorithm (Algorithm 2); the proof then consists of showing that the greedy algorithm is well defined and returns the maximal tableau.

1. Define the *canonical maximal strings* $\mathbf{S}_{\max}(\tau)$.
2. Prove that Algorithm 2 yields the unique maximal tableau $T_{\max}[\tau]$ with Temperley–Lieb pattern $\mathbf{S}_{\max}(\tau)$.
   (i) Show that Algorithm 2 is well defined.
   (ii) Prove that Algorithm 2 produces the unique maximal Yamanouchi tableau with Temperley–Lieb pattern $\mathbf{S}_{\max}(\tau)$.
3. Prove that if a Yamanouchi tableau T that has Temperley–Lieb pattern different from $\mathbf{S}_{\max}(\tau)$, then one can construct a tableau of type τ with larger content.

Example 5.1. Let $\lambda = (9, 7, 6, 4)$, $\mu = (4, 3, 1, 0)$, R be the $1\times\infty$ ribbon, and $\tau = \{\{P_1, P_2\}, \{P_3, P_4\}, \{Q_1, Q_2\}, \{Q_3, Q_4\}\}$. Algorithm 2 returns the following tableau of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and type τ; the Temperley–Lieb pattern of $\mathbf{S}_{\max}(\tau)$ is also shown.



Consider the square $\square$ containing the bolded red entry labeled **4**. In the step of filling $\square$ in Algorithm 2, we determine that the set of possible values must be the intersection of:

- $\{1, 2, 3, 4\}$, from row monotonicity;
- $\{1, 2, \dots\}$, from column monotonicity;
- $\{3, 4, \dots\}$, from the horizontal strings between $\square$ and $\square + (1, -1)$;
- $\{1, 2, 3, 4, 5\}$, from the Yamanouchi condition.

The intersection of the above sets is $\{3, 4\}$, which is nonempty. In this step of the algorithm, we choose the largest element in $\{3, 4\}$, which is 4.

Similar to Section 4, we define the maximal canonical strings $\mathbf{S}_{\max}(\tau)$ to be the unique Temperley–Lieb pattern satisfying two pattern avoidances: (UR) and (⊔). We construct a map φ on Temperley–Lieb patterns containing the sub-pattern (UR) to delete the last (UR) sub-pattern while preserving Temperley–Lieb type (Lemma 5.7); we construct a similar, though simpler map ϕ which does the same for (⊔) sub-patterns. Using alternating applications of φ and ϕ, we construct a Temperley–Lieb pattern of type τ satisfying the desired pattern avoidances, and establish uniqueness separately (Proposition 5.12).

Following this, we define the filling $T_{\max}[\tau]$ via a greedy algorithm, and prove that it is a Yamanouchi tableau (Corollary 5.15). Our greedy construction, more generally, gives us a way to automatically fill the remainder of a partially filled tableaux so that it is Yamanouchi (Proposition 5.14). We then show that the greedy algorithm returns elementwise maximal tableaux among the Yamanouchi tableaux with fixed Temperley–Lieb pattern (Lemma 5.16).

The remaining task is then to prove the maximality of the canonical maximal strings. This is equivalent to constructing, for tableaux T with Temperley–Lieb pattern $\mathbf{S} \neq \mathbf{S}_{\max}(\tau)$, a tableaux T' of type τ and content strictly smaller than τ. We note that the φ construction is given by a set of orientation toggles on various squares; these squares can be partitioned into *Type* $(A, \mathbf{S})$ *sets* of size 1 and *Type* $(B, \mathbf{S})$ sets of size 2 (Lemma 5.9). To construct T', we use a local tableau construction which toggles the orientations of the squares in the last Type $(A, \mathbf{S})$ or Type $(B, \mathbf{S})$ set in the Temperley–Lieb pattern while satisfying the Yamanouchi condition (Proposition 5.18). Repeatedly applying the local tableau construction will eventually yield a Temperley–Lieb pattern of $\varphi(\mathbf{S})$ or $\varphi(\phi(\mathbf{S}))$, which completes the construction of T'.

5.2 Canonical maximal strings

Definition 5.2. Given a shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and Temperley–Lieb type τ, we define the *canonical maximal strings* $\mathbf{S}_{\max}(\tau)$ to be the unique Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with type τ which avoids the two patterns below.



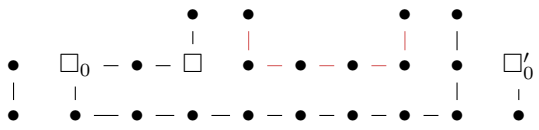
Lemma 5.3. *Suppose that the Temperley–Lieb pattern* $\mathbf{S}$ *on* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *contains no* (UR) *sub-patterns. For every square* $\square \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$*, there is an endpoint* $R \in \{P_i, Q_i : 1 \leq i \leq \ell\}$ *such that* $\square + (1, 0)$ *and* R *are connected by an up-right lattice path in* $\mathbf{S}$*. In particular,* $\mathbf{S}$ *is acyclic.*

Proof. It suffices to prove that for any square $\square \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, the point $p := \square + (1,0)$ satisfies the property that one of the pairs $(p, p+(0,1))$ and $(p, p+(1,0))$ is connected in $\mathbf{S}$. If the line segments for the pair $(\square, \square+(1,1))$ are vertical, then $(p, p+(0,1))$ is connected in $\mathbf{S}$. Otherwise, the line segments for $(\square, \square+(1,1))$ are horizontal, so by the avoidance of (UR) the line segments for $(\square+(1,-1), \square+(2,0))$ are horizontal. In this case, the pair $(p, p+(1,0))$ is connected in $\mathbf{S}$, completing the proof. ■

Definition 5.4. Consider a Temperley–Lieb pattern $\mathbf{S}$. We say that a square $\square$ is a *(UR)-corner* in $\mathbf{S}$ if $\square$ has horizontal orientation and $\square + (1,-1)$ has vertical orientation. We say that $\square$ is a $(\sqcup)$*-corner* in $\mathbf{S}$ if $\square$ is the left endpoint of a $(\sqcup)$ pattern. A $(\sqcup)$-corner $\square$ with corresponding right endpoint $\square' + (-1,0)$ is *primitive* if $\square$ and $\square' + (-2,0)$ are connected by a horizontal path in $\mathbf{S}$.

Lemma 5.5. *Let $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ which has at least one $(\sqcup)$ sub-pattern and no (UR) sub-patterns. Then the minimal $(\sqcup)$-corner of $\mathbf{S}$ is primitive.*

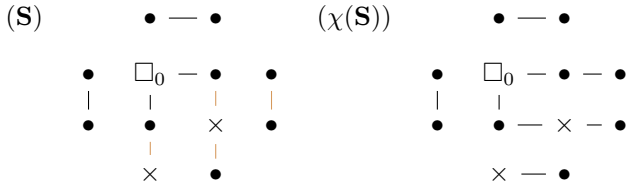
Proof. Let $\square_0$ be the minimal $(\sqcup)$-corner, and let $\square_0' + (-1,0)$ be the right endpoint of the corresponding $(\sqcup)$ sub-pattern. Assume for contradiction that $\square_0$ is not primitive. By the avoidance of (UR), the orientation of $\square_0' + (-2,0)$ is vertical. Let $\square$ be the smallest square greater than $\square_0' + (-2,0)$ with vertical orientation; note that $\square > \square_0$, or else $\square_0$ is primitive. Then there is a $(\sqcup)$ sub-pattern with endpoints at $\square + (1,1)$ and $\square_0' + (0,1)$. Thus, $\square + (1,1) < \square_0$ is a $(\sqcup)$-corner, contradicting the minimality of $\square_0$.



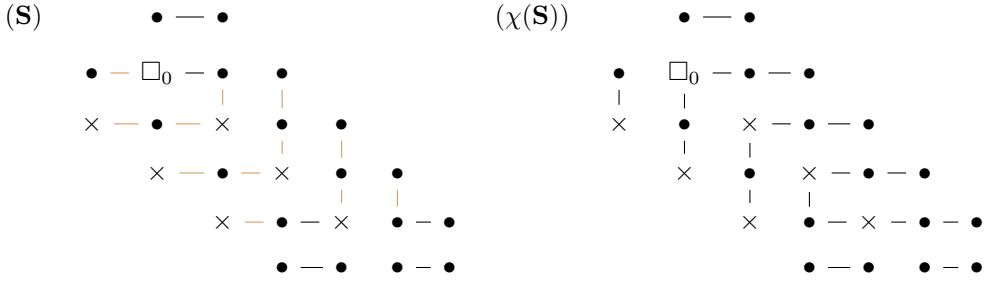
■

Definition 5.6. Let $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ which has at least one (UR) sub-pattern. We will define the Temperley–Lieb pattern $\varphi(\mathbf{S})$ as follows. Let $\square_0$ denote the greatest square such that $\square_0$ is a (UR)-corner. Let r be the greatest positive integer such that the orientation of $\square_0 + (k,-k)$ is vertical for each $1 \le k \le r$. We define the *toggling set* $\mathsf{T}(\mathbf{S})$ by the casework below; subsequently, we define $\chi(\mathbf{S})$ by taking $\mathbf{S}$ and toggling the orientation of each square in $\mathsf{T}(\mathbf{S})$.

- Case 1: Suppose the orientation of $\square_0 + (0,-2)$ is vertical. Then, set $\mathsf{T}(\mathbf{S}) := \{\square_0 + (0,-2), \square_0 + (1,-1)\}$.



- Case 2: Suppose the orientation of $\square_0 + (0,-2)$ is horizontal. If the orientation of $\square_0 + (-1,-1)$ is horizontal, then set $\mathsf{T}(\mathbf{S}) := \{\square_0 + (1,-1), \ldots, \square_0 + (r,-r)\} \sqcup \{\square_0 + (-1,-1), \ldots, \square_0 + (r-2,-r)\}$. Otherwise, set $\mathsf{T}(\mathbf{S}) := \{\square_0 + (1,-1), \ldots, \square_0 + (r,-r)\} \sqcup \{\square_0 + (0,-2), \ldots, \square_0 + (r-2,-r)\}$.

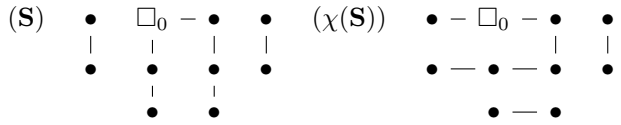


Let j denote the largest positive integer such that the greatest (UR)-corner of $\chi^k(\mathbf{S})$ is greater than the greatest (UR)-corner of $\chi^{k-1}(\mathbf{S})$ for each $1 \le k \le j-1$. We define $\chi_{\mathrm{tot}}(\mathbf{S}) := \chi^j(\mathbf{S})$, and $\varphi(\mathbf{S}) := \chi^m_{\mathrm{tot}}(\mathbf{S})$, where m is the number of (UR)-corners in $\mathbf{S}$. Note that $\varphi(\mathbf{S})$ has no (UR) sub-patterns.

Lemma 5.7. *The map φ preserves Temperley–Lieb type.*

Proof. It suffices to check that χ is type-preserving. We split into cases on the orientation of $\square_0 + (1, -1)$.

Case 1: By the maximality of $\square_0$, the orientation of $\square_0 + (-1, -1)$ must be vertical. Then, we have the following diagram, which proves type-invariance for this case.



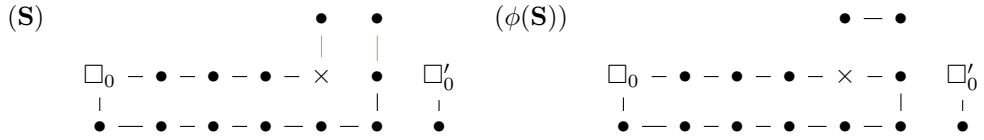
Case 2: Note that for each $2 \leq k \leq r$, the orientation of $\square_0 + (k-2, -k)$ must be horizontal, or else $\square_0 + (k-1, -k-1) > \square_0$ is a (UR)-corner, which contradicts the maximality of $\square_0$. Now, observe that for each $2 \leq k \leq r$, the squares $\square_0 + (k-2, -k)$ and $\square_0 + (k+1, -k+1)$ are connected in $\mathbf{S}$. Since this connectivity holds in $\chi(\mathbf{S})$, we have that χ is type-preserving in this case as well, completing the proof. ■

Definition 5.8. A *Type* $(A, \mathbf{S})$ *set* is a set of the form $\{\square\} \subseteq (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, where $\square$ is of vertical orientation. A *Type* $(B, \mathbf{S})$ *set* is a set of the form $\{\square, \square + (2m, 0)\} \subseteq (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, where m is a positive integer, $\square + (2k, 0)$ is of horizontal orientation for all $0 \leq k \leq m-1$, and $\square + (2m, 0)$ is of vertical orientation.

Lemma 5.9. *Let $\triangle$ denote the symmetric difference operator on sets. The set $\triangle_{i \geq 0} \mathsf{T}(\chi^i(\mathbf{S}))$ can be partitioned into Type $(A, \mathbf{S})$ sets and Type $(B, \mathbf{S})$ sets.*

Proof. Let $\square \in \triangle_{i \geq 0} \mathsf{T}(\chi^i(\mathbf{S}))$ such that $\square$ has horizontal orientation in $\mathbf{S}$. Let j be the greatest positive integer such that $\square + (2k, 0) \in \triangle_{i \geq 0} \mathsf{T}(\chi^i(\mathbf{S}))$ for all $1 \leq k \leq j$. Then, $\square + (2j, 0)$ has vertical orientation in $\mathbf{S}$ as $\square + (2j+2, 0) \notin \triangle_{i \geq 0} \mathsf{T}(\chi^i(\mathbf{S}))$, so $\{\square, \square + (2j, 0)\}$ is a Type $(B, \mathbf{S})$ set. Thus, the desired result follows. ■

Definition 5.10. Let $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with no (UR) sub-pattern. Let $\square_0$ denote the minimal $(\sqcup)$-corner in $\mathbf{S}$, and let $\square_0' + (-1, 0)$ be the right endpoint of the $(\sqcup)$ sub-pattern containing $\square_0$. We define $\phi(\mathbf{S})$ to be $\mathbf{S}$ with the orientation of $\square_0' + (-2, 0)$ toggled.



Lemma 5.11. *Let $\mathbf{S}$ be a Temperley–Lieb pattern. Then the minimal $(\sqcup)$-corner of $(\varphi \circ \phi)(\mathbf{S})$ is greater than or equal to the minimal $(\sqcup)$-corner of $\mathbf{S}$.*

Proof. We split into cases on the existence of $(\sqcup)$-corners in $\mathbf{S}$.

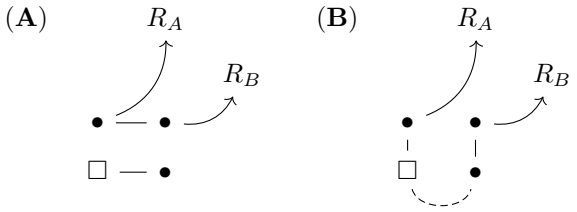
Case 1: Suppose there is at least one $(\sqcup)$-corner in $\mathbf{S}$. Let $\square_0$ denote the minimal $(\sqcup)$-corner in $\mathbf{S}$. It follows that $\square_0$ is not an $(\sqcup)$-corner in $(\varphi \circ \phi)(\mathbf{S})$ and no $(\sqcup)$-corner in $(\varphi \circ \phi)(\mathbf{S})$ is less than $\square_0$, completing the proof for this case.

Case 2: Suppose $\mathbf{S}$ has no $(\sqcup)$-corners. Assume for contradiction that $\varphi(\mathbf{S})$ has at least one $(\sqcup)$ sub-pattern. Since $\varphi(\mathbf{S})$ has no (UR) sub-patterns, it follows by Lemma 5.5 that $\varphi(\mathbf{S})$ has a primitive $(\sqcup)$-corner $\square_0$. Since neither $\square_0$ nor $\square_0'$ are in $\triangle_{i \geq 0} \mathsf{T}(\chi^i(\mathbf{S}))$, then the cup containing $\square_0$ is in $\mathbf{S}$, a contradiction. Hence, the proof is complete for this case. ■

Proposition 5.12. *Suppose that τ is a Temperley–Lieb type on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ such that $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$. Then, there exists a unique Temperley–Lieb pattern $\mathbf{S}_{\max}(\tau)$ on the shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ with type τ such that $\mathbf{S}$ avoids the patterns from Definition 5.2. That is, Definition 5.2 is well defined.*

Proof. Since $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$, there exists a Temperley–Lieb pattern $\mathbf{S}_0$ on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$. By Lemma 5.11, we have that for each Temperley–Lieb pattern $\mathbf{S}$ for which $(\varphi \circ \phi)(\mathbf{S})$ is defined, the minimal $(\sqcup)$-corner of $(\varphi \circ \phi)(\mathbf{S})$ is greater than the minimal $(\sqcup)$-corner of $\mathbf{S}$. Thus, there exists a nonnegative integer N such that N is the maximal integer for which $((\varphi \circ \phi)^N \circ \varphi)(\mathbf{S})$ is defined. By the maximality of N, the Temperley–Lieb pattern $((\varphi \circ \phi)^N \circ \varphi)(\mathbf{S})$ avoids (UR) and $(\sqcup)$ patterns, completing the proof of existence.

Next, we prove that there is at most one Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ of Temperley–Lieb type τ which avoids (UR) and (⊔). Assume for contradiction that there exist Temperley–Lieb patterns $\mathbf{A} \neq \mathbf{B}$ on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ of the same type that both avoid the patterns (UR) and (⊔). Let $\square$ be the minimal square in $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ such that orientation for $\square$ in $\mathbf{A}$ and the orientation for $\square$ in $\mathbf{B}$ differ; assume without loss of generality that these are horizontal and vertical, respectively. Note that the hypotheses of Lemma 5.3 apply to $\square + (1,0)$ and $\square + (3,0)$, or else we have a contradiction to the avoidances of (U) and (R). By Lemma 5.3, there exist endpoints $R_A, R_B \in \{P_i, Q_i : 1 \leq i \leq \ell\}$ such that $\square + (1,0)$ is connected to R_A by an up-right lattice path and $\square + (3,0)$ is connected to R_B by an up-right lattice path (note that these statements hold in both $\mathbf{A}$ and $\mathbf{B}$ by the minimality of $\square$). Since $\{R_A, R_B\}$ is connected in $\mathbf{A}$, it must be connected in $\mathbf{B}$ as well since $\mathbf{A}$ and $\mathbf{B}$ have the same Temperley–Lieb type. In particular, the pair $(\square, \square + (1,0))$ is connected in $\mathbf{B}$ by a path $\square = p_1 \to \cdots \to p_k = \square + (1,0)$.



Consider the sequence $(v_1, \ldots, v_{k-1}) := (p_2 - p_1, \ldots, p_k - p_{k-1})$. Note that $v_i \in \{(0, \pm 1), (\pm 1, 0)\}$ for every index i, and that there exist indices $i_- < i_+$ such that

- $v_{i_-} \in \{(0, \pm 1)\}$;
- $v_{i_+} = \{(0, \pm 1)\} \setminus \{v_{i_-}\}$;
- either $v_j = (0,1)$ for all $j \in (i_-, i_+)$ or $v_j = (0,-1)$ for all $j \in (i_-, i_+)$.

Observe that $v_{i_-} = (0,-1)$ and $v_{i_+} = (0,1)$, or else the path $p_{i_+-1} \to p_{i_+} \to p_{i_++1}$ contradicts the avoidance of (UR). In this case, the path $p_{i_-} \to \cdots \to p_{i_++1}$ is a contradiction to the avoidance of (⊔), completing the proof of uniqueness. ■

5.3 Construction

We will construct the maximal tableau $T_{\max}[\tau]$ from the following greedy algorithm.

Definition 5.13. Let $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be a shuffle shape, $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, and w be a word of length $|w| \leq |(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)|$. Define $\mathsf{Next}(\mathbf{S}, w)$ to be the set of possible values for the $(|w|+1)$th square in the reading order for which the tableau conditions, Temperley–Lieb pattern condition, and word-Yamanouchi condition persist.

Algorithm 2. *Given a shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, a Temperley–Lieb pattern $\mathbf{S}$ on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and a partial word w of $\mathbf{S}$, we define the filling $T_{\max}(\mathbf{S}, w)$ of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and Temperley–Lieb pattern $\mathbf{S}$ by the following process.*

- *Start with the $(|w|+1)$th square in the reading order.*
- *Fill each square $\square$ with* $\max(\mathsf{Next}(\mathbf{S}, \mathsf{w}(\square)))$.

We denote the filling $T_{\max}(\mathbf{S}_{\max}(\tau), \varepsilon)$ by $T_{\max}[\tau]$.

Proposition 5.14. *Let $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be a shuffle shape, $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, and w be a partial word of $\mathbf{S}$. Suppose that no square greater than the $|w|$th square in the reading order is a* (UR)*-corner of $\mathbf{S}$. Let $\square$ be the $(|w|+1)$th square in the reading order of $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, and suppose that $\square$ is not a bottleneck of $\mathbf{S}$. Then $\mathsf{Next}(\mathbf{S}, w)$ is non-empty.*

Proof. Consider the diagram below.

c

b

$\square$ a

Assume for contradiction that no positive integer can be filled in $\square$ to satisfy the row monotonicity, column monotonicity, string inequality, and word-Yamanouchi conditions. We perform casework on the orientation of $\square$.

Case 1: Suppose that the orientation of $\square$ is horizontal. Then, for all integers $k \in [b, a)$, the word $\mathsf{w}(\square+(2,0))$ is $(k, k+1)$-balanced. We claim that $a = b+1$. By Lemma 3.2, we have $a \geq b+1$. Suppose that $a \geq b+2$. Given an integer x that appears in the tableau, let $\mathsf{t}(x, \square_0)$ denote the greatest square $\leq \square_0$ containing x. Note that if $\mathsf{w}(\mathsf{t}(x+1)+(-2,0), \square)$ is $(x, x+1)$-balanced, then $\mathsf{t}(x+1)$ is at least one row below $\mathsf{t}(x)$. Thus, $\mathsf{t}(b+2)$ is at least two rows below $\mathsf{t}(b)$, a contradiction. This implies that $a \leq b+1$, confirming the equality $a = b+1$. However, by Lemma 3.8, this implies that the Temperley–Lieb pattern up to this point is inconsistent with $\mathbf{S}$, a contradiction.

Case 2: Suppose that the orientation of $\square$ is vertical. Then, by the avoidance of (UR), the entry in $\square+(-1,1)$ is $\leq c$. Note that if $\square+(-1,1)$ has entry c, then the last occurrence of c in $\mathsf{w}(\square)$ appears after the last occurrence of $c+1$, implying that $\mathsf{w}(\square)$ is not $(c, c+1)$-balanced, a contradiction. Thus, $\square+(-1,1)$ has entry $\leq c-1$, so $\mathsf{w}(\square+(-1,1))$ is $(c, c+1)$-balanced. By Lemma 3.8, this means that the Temperley–Lieb pattern of the tableaux is not $\mathbf{S}$, a contradiction.

Hence $T_{\max}(\mathbf{S}, w)$ is word-Yamanouchi, so by Proposition 3.3, $T_{\max}(w, \mathbf{S})$ is Yamanouchi, as desired. ■

Corollary 5.15. *The filling $T_{\max}[\tau]$ is a Yamanouchi tableau.*

Proof. Since $\mathbf{S}_{\max}(\tau)$ contains no (UR) sub-patterns, $T_{\max}[\tau]$ is a Yamanouchi tableau by Proposition 5.14 with $\mathbf{S} := \mathbf{S}_{\max}(\tau)$ and $w := \varepsilon$. ■

In this subsection, we prove that $T_{\max}[\tau]$ is the maximal Yamanouchi tableau of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and type τ.

Lemma 5.16. *Let $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be a shuffle shape, $\mathbf{S}$ be a Temperley–Lieb pattern on $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$, and w be a partial word of $\mathbf{S}$. Suppose that no square greater than the $|w|$th square in the reading order is a* (UR)*-corner of $\mathbf{S}$. Then $T_{\max}(\mathbf{S}, w)$ is the elementwise maximal Yamanouchi tableau with reading word prefix w and Temperley–Lieb pattern $\mathbf{S}$.*

Proof. Let $T \neq T_{\max}(\mathbf{S}, w)$ be a Yamanouchi tableau with reading word prefix w and Temperley–Lieb pattern $\mathbf{S}$. Let $\square_0 \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ be the maximal square such that the entry a of $\square_0$ in T is strictly less than the entry b of $\square_0$ in $T_{\max}(\mathbf{S}, w)$. Note that $\square_0$ is greater than the $|w|$th square in the reading order. Let T_0 denote the filling given by T with the entry of $\square_0$ replaced by b. We claim that in T_0, the diagonal inequalities for $\mathbf{S}$ and column monotonicity are satisfied.

- If the orientation of $\square_0+(-1,-1)$ is horizontal, then the diagonal inequality for $\square_0+(-1,-1)$ remains valid as $b > a$.
- Suppose that the orientation of $\square_0+(-1,-1)$ is vertical. By Lemma 3.2 applied to $T_{\max}(\mathbf{S}, w)$, the entry of $\square_0+(1,-1)$ in T_0 is $\geq b+1$. Since placing an entry of $b+1$ in $\square_0+(-1,-1)$ satisfies the word-Yamanouchi condition at $\square_0+(-1,1)$, the entry of $\square_0+(-1,-1)$ in T_0 is $\geq b+1$. Thus, the diagonal inequality for $\square_0+(-1,-1)$ remains valid in this case as well.
- Since placing an entry of $b+1$ in $\square_0+(0,-2)$ satisfies the word-Yamanouchi condition at $\square_0+(0,-2)$, the entry of $\square_0+(0,-2)$ in T_0 is $\geq b+1$, so column monotonicity holds in T_0.

Now, we will construct a Yamanouchi tableau T' with Temperley–Lieb pattern $\mathbf{S}$ that is elementwise greater than T_0. It suffices to construct T' so that T' is word-Yamanouchi, all squares up to and including $\square_0$ in the reading order have the same entry in T' as in T_0, and T_0 is elementwise greater than T_0. We perform the following process to obtain T'.

1. For $k \geq 0$ (starting with $k = 0$), perform the following:
   - If the reading word of the current tableau is not $(a+k, a+k+1)$-balanced in some prefix, then increment the entry of the minimal square greater than $\square_0$ with entry $a+k+1$ by 1.
   - Otherwise, terminate.

2. If $b = a + 1$, then terminate. Else, for $k' \geq 0$ (starting with $k' = 0$), perform the following:

   - If the reading word of the current tableau is not $(b + k - 1, b + k)$-balanced in some prefix, then increment the entry of the minimal square greater than $\square_0$ with entry $b + k$ by 1.
   - Otherwise, terminate.

Observe that T' generated by the above process satisfies all of the desired properties. It remains to verify that T' has Temperley–Lieb pattern $\mathbf{S}$. Assume for contradiction that for some x, the minimal square $\square > \square_0$ such that $\square$ has entry x changes orientation upon being incremented by 1. Then, the entry in $\square + (1, 1)$ must be x, which is a contradiction to the minimality of $\square$ as $\square > \square + (1, 1) > \square_0$. Thus, T' has Temperley–Lieb pattern $\mathbf{S}$, completing the proof.

■

Now, we turn to proving the optimality of the canonical maximal strings $\mathbf{S}_{\max}(\tau)$. We first prove a fact about the maximal element of the toggle set associated with φ.

Lemma 5.17. *Suppose* $\{\square > \square'\} \subseteq \triangle_{i \geq 0}\mathsf{T}(\chi^i(\mathbf{S}))$ *is a Type* $(B, \mathbf{S})$ *set such that* $\square = \max\left(\triangle_{i \geq 0}\mathsf{T}(\chi^i(\mathbf{S}))\right)$ *and* $\square + (-2, 0) \in (\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$. *Then* $\square + (-2, 0)$ *is not a bottleneck of* $\mathbf{S}$.

Proof. Assume for contradiction that $\square + (-2, 0)$ is a bottleneck of $\mathbf{S}$.

We begin by claiming that the entry in $\square'$ is $x + 1$. Assume otherwise. Let k be the largest nonnegative integer such that $\square + (2k, 0)$ has entry $x + 1$. Since the entry in $\square + (2k + 1, 1)$ is weakly bounded below by x (from row monotonicity) and weakly bounded above by x (since the orientation of $\square + (2k, 0)$ is horizontal), the entry in $\square + (2k + 1, 1)$ is exactly x. By the maximality of k, the word $\mathsf{w}(\square + (2k + 3, 1), \square + (2k + 2, 0))$ contains at least one more occurrence of x than of $x + 1$, which implies that $\mathsf{w}(\square)$ is not $(x, x + 1)$-balanced, a contradiction. Thus, the entry in $\square'$ is $x + 1$.

Next, we claim that the entry in $\square' + (0, -2)$ is less than x. Assume for the contradiction that the entry in $\square' + (0, -2)$ is x; then, we can remove the $(x, x + 1)$-overlapping pair $(\square' - (0, 2), \square')$ from $\mathsf{w}(\square)$ and still obtain a lattice word. However, upon performing this removal, the suffix $\mathsf{w}(\square' + (1, 1), \square)$ contains at least one more occurrence of x than of $x + 1$, a contradiction. Hence, the entry in $\square' + (0, -2)$ is less than x.

Note that the second claim implies that $\square' + (-1, 1)$ is a (UR)-corner, so $\triangle_{i \geq 0}\mathsf{T}(\chi^i(\mathbf{S})) = \{\square > \square'\}$ and $\varphi(\mathbf{S})$ can be obtained by toggling the orientations of $\square$ and $\square'$. However, upon doing so, we see that $\square + (-1, -1)$ is a (UR)-corner of $\varphi(\mathbf{S})$, a contradiction. Therefore, $\square + (-2, 0)$ is not a bottleneck of $\mathbf{S}$. ■

Proposition 5.18. *Let* T *be a tableau of shape* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *and Temperley–Lieb type* τ *with Temperley–Lieb pattern different from* $\mathbf{S}_{\max}(\tau)$. *Then, there exists a Yamanouchi tableau of shape* $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ *and Temperley–Lieb type* τ *with content strictly smaller than* $\lambda(T)$.

Proof. Let $\mathbf{S}$ denote the Temperley–Lieb pattern of T; we will construct a Yamanouchi tableau with Temperley–Lieb pattern $\varphi(\mathbf{S})$, which would complete the proof as φ is type-preserving by Lemma 5.7.

Denote by $\square_0$ the greatest element of $\triangle_{i \geq 0}\mathsf{T}(\chi^i(\mathbf{S}))$. Note that $\square_0$ lies in either a Type $(A, \mathbf{S})$ set or a Type $(B, \mathbf{S})$ set. If $\square_0$ lies in a Type $(A, \mathbf{S})$ set, then let $\mathbf{S}'$ be the Temperley–Lieb pattern obtained by taking $\mathbf{S}$ and toggling the orientation of $\square_0$; if $\square_0$ lies in a Type $(B, \mathbf{S})$ set, then let $\mathbf{S}'$ be the Temperley–Lieb pattern obtained by taking $\mathbf{S}$ and toggling the orientations of both $\square_0$ and $\square_0'$, where $\square_0'$ is the other element in the Type $(B, \mathbf{S})$ set containing $\square_0$. We will show that in both cases, $T_{\max}(\mathbf{S}', \mathsf{w}(\square_0))$ is elementwise greater than T.

Case 1: $\square_0$ lies in a Type $(A, \mathbf{S})$ set. Since toggling the orientation of $\square$ does not introduce any (UR) sub-patterns, the filling $T_{\max}(\mathbf{S}', \mathsf{w}(\square_0))$ is a Yamanouchi tableau with Temperley–Lieb pattern $\mathbf{S}'$, by Corollary 5.15. Thus, by Lemma 5.16, $T_{\max}(\mathbf{S}', \mathsf{w}(\square_0))$ is elementwise greater than T.

Case 2: $\square_0$ lies in a a Type $(B, \mathbf{S})$ set $\{\square_0, \square_0 + (2m, 0)\}$, where m is a positive integer. For each $0 \leq i \leq m$, let a_i and b_i denote the entries in $\square_0 + (2i + 1, 1)$ and $\square_0 + (2i, 0)$, respectively. We will perform the following operation on the entries of the squares in $\{\square_0 + (2i, 0), \square_0 + (2i + 1, 1) : 0 \leq i \leq m\}$. Let I denote the set of all indices $0 \leq i \leq m$ for which $b_i < b_{i+1}$. For each $i \in I$, change the entry in $\square_0 + (2i + 1, 1)$ to b_{i+1}, and for each $1 \leq j \leq m$, change the entry in $\square_0 + (2i + 1, 1)$ to b_i and the entry in $\square_0 + (2i, 0)$ to $b_i + 1$. Let w' denote the updated reading word up to $\square$. By Lemma 5.17, Proposition 5.14, and Lemma 5.16, the filling $T_{\max}(\mathbf{S}', \mathsf{w}(\square_0))$ is a Yamanouchi tableau with strictly smaller content than T.

Repeatedly applying the operation $T \mapsto T_{\max}(\mathbf{S}', \mathsf{w}(\square_0))$ eventually returns a Yamanouchi tableau with Temperley–Lieb pattern $\varphi(\mathbf{S})$ and strictly smaller content than T, completing the proof. ■

5.4 Proof of Theorem 1.3, Part (ii)

Now, we may conclude the proof of Theorem 1.3.

Proof of Theorem 1.3, Part (ii). By Corollary 5.15, $T_{\max}[\tau]$ is the unique maximal Yamanouchi tableau among those with Temperley–Lieb pattern $\mathbf{S}_{\max}(\tau)$. Since all Yamanouchi tableaux with Temperley–Lieb type τ and Temperley–Lieb pattern different from $\mathbf{S}_{\max}(\tau)$ have content strictly greater than $\lambda(T_{\max}[\tau])$ by repeated application of Proposition 5.18, we conclude that $T_{\max}[\tau]$ is the unique maximal tableau of shape $(\lambda_R/\mu_R) \circledast (\lambda_B/\mu_B)$ and Temperley–Lieb type τ. ■

6 Further Exploration and Open Problems

6.1 Specialization of Jacobi–Trudi matrices

We briefly consider the special case of the Schur supports of Temperley–Lieb immanants of Jacobi–Trudi matrices. For this class of matrices, we offer an algebraic proof of the existence of a unique minimal partition in the Schur support, which comes with a clean description of the minimal partition.

For strict partitions μ and ν of length n, we define the *Jacobi–Trudi matrix* $A_{\mu,\nu}$ by $(h_{\mu_i-\nu_j})_{i,j=1}^n$. For 321-avoiding permutations w, the Temperley–Lieb immanant $\mathrm{Imm}_{t_w}(A_{\mu,\nu})$ is thus given by

$$\mathrm{Imm}_{t_w}(A_{\mu,\nu}) = \sum_{v\in S_n} f_w(v) \prod_{i=1}^{n} h_{\mu_i-\nu_{w(i)}} = \sum_{v\in S_n} f_w(v)\cdot h_{\mathrm{sort}(\mu_i-\nu_{w(i)})}.$$

Theorem 6.1. *Let μ and ν be strict partitions of length ℓ and $\tau = t_w$ be a basis element of $\mathrm{TL}_\ell(2)$ with $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) \neq 0$, where w is a 321-avoiding permutation in S_ℓ. Then the partition $\mathrm{sort}(\mu_i - \nu_{w(i)})$ is the unique minimal element of $\mathrm{Supp}_s(\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}))$ with respect to the dominance order, and appears with coefficient 1.*

Proof. Let $\leq_B$ denote the Bruhat order. We claim that for any two permutations $v, v' \in S_\ell$ with $v >_B v'$, one has

$$\mathrm{sort}(\mu_i - \nu_{v(i)}) >_D \mathrm{sort}(\mu_i - \nu_{v'(i)}).$$

It suffices to verify the above in the case where $v = v'\circ(i_1\ i_2)$ for indices $i_1 < i_2$ such that $\nu_{v(i_1)} < \nu_{v(i_2)}$ and $\nu_{v'(i_1)} > \nu_{v'(i_2)}$. For integers $k \in [1,\ell]$, define the partial sums D_k and D'_k by

$$\begin{cases} D_k := \sum_{m=1}^{k} (\mathrm{sort}(\mu_i - \nu_{v(i)}))_m \\ D'_k := \sum_{m=1}^{k} (\mathrm{sort}(\mu_i - \nu_{v'(i)}))_m \end{cases}.$$

Let j_1 and j_2 denote the positions of $\mu_{i_1}-\nu_{v(i_1)}$ and $\mu_{i_2}-\nu_{v(i_2)}$ in $\mathrm{sort}(\mu_i-\nu_{v(i)})$, respectively. Observe that for indices $k \notin [j_1, j_2-1]$, the equality $D_k = D'_k$ holds. For $k \in [j_1, j_2-1]$, we have $D_k - D'_k > 0$. As such, $D_k \geq D'_k$ for every index k, with strict inequality at $k = j_1$. We conclude that $\mathrm{sort}(\mu_i - \nu_{v(i)}) >_D \mathrm{sort}(\mu_i - \nu_{v'(i)})$, as claimed.

Note that for every permutation $v \in S_\ell$ with $f_w(v) \neq 0$, we have $v \geq_B w$. Thus,

$$\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) = \sum_{v\geq_B w} f_w(v)\cdot h_{\mathrm{sort}(\mu_i-\nu_{v(i)})}.$$

Now, we verify that $\mu_i - \nu_{w(i)} \geq 0$ for each index $i \in [1,\ell]$. Assume for contradiction that $\min_i(\mu_i-\nu_{w(i)}) < 0$. By the claim, $\min_i(\mu_i - \nu_{w(i)}) \leq \min_i(\mu_i - \nu_{v(i)}) < 0$ for all v with $v \geq_B w$, so it follows that

$$\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) = \sum_{v\geq_B w} f_w(v)\cdot 0 = 0,$$

a contradiction. Thus, $\mu_i - \nu_{w(i)} \geq 0$ for each index $i \in [1,\ell]$.

Since h_λ is Schur-positive with $\min(\mathrm{Supp}_s(h_\lambda)) = \lambda$ and $[s_\lambda]h_\lambda = 1$ for every partition λ, it follows by the claim that $\mathrm{sort}(\mu_i - \nu_{w(i)})$ is the unique minimal element of $\mathrm{Supp}_s(\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}))$ with respect to the dominance order and appears with coefficient $f_w(w) = 1$, as desired. ■

Remark 6.2. The key ideas in the proof of Theorem 6.1 fail to extend to ribbon decomposition matrices. More broadly, there does not appear to be a similar clean description of the minimal partition in the generality of ribbon decomposition matrices.

We now discuss a log-concavity conjecture for Temperley–Lieb immanants of Jacobi–Trudi matrices. The first log-concavity conjecture in the study of Schur-positivity was the Lam–Postnikov–Pylyavskyy (LPP) conjecture [LPP07], which asserts that Schur functions satisfy a log-concavity inequality in the Schur basis. After a long run of attempts via tableaux-based methods, the LPP conjecture was proved by Speyer in [Spe26] using a new polyhedral model for Littlewood–Richardson coefficients called *skeps*. In [LN26], Le and Nguyen introduced the *skew skep* model, which was used to prove the following skew Schur log-concavity result.

Definition 6.3. Given two integer tuples λ, μ, the parallelepiped $P_{\lambda,\mu}$ is the minimal alcoved polytope containing λ and μ.

Theorem 6.4 ([LN26], Theorem 1.2)**.** *Let $\lambda, \mu, \nu, \rho, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}, \widetilde{\rho}$ be partitions, not necessarily ending in 0. Denote $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$. If $(\lambda, \mu) + (\nu, \rho) = (\widetilde{\lambda}, \widetilde{\mu}) + (\widetilde{\nu}, \widetilde{\rho})$, and $(\widetilde{\lambda}, \widetilde{\mu}), (\widetilde{\nu}, \widetilde{\rho}) \in P_{(\lambda,\mu),(\nu,\rho)}$, then $s_{\widetilde{\lambda}/\widetilde{\mu}} s_{\widetilde{\nu}/\widetilde{\rho}} - s_{\lambda/\mu} s_{\nu/\rho}$ is Schur-positive.*

We conjecture that [LN26, Theorem 1.2] generalizes to Temperley–Lieb immanants of Jacobi–Trudi matrices as follows. Note that setting τ to the trivial Temperley–Lieb type would return [LN26, Theorem 1.2].

Conjecture 6.5. *Let $\lambda, \mu, \nu, \rho, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}, \widetilde{\rho}$ be partitions satisfying the hypotheses of Theorem 6.4. Then*

$$\operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\widetilde{\lambda}^*,\widetilde{\mu}^*}) \operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\widetilde{\nu}^*,\widetilde{\rho}^*}) \geq_s \operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\lambda^*,\mu^*}) \operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\nu^*,\rho^*}).$$

Inspired by [DP07], we propose a weaker support-based version of Conjecture 6.5 below.

Conjecture 6.6. *Let $\lambda, \mu, \nu, \rho, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}, \widetilde{\rho}$ be partitions satisfying the hypotheses of Theorem 6.4. Then*

$$\operatorname{Supp}_s \left(\operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\lambda^*,\mu^*}) \operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\nu^*,\rho^*}) \right) \subseteq \operatorname{Supp}_s \left(\operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\widetilde{\lambda}^*,\widetilde{\mu}^*}) \operatorname{Imm}_\tau^{\mathrm{TL}}(A_{\widetilde{\nu}^*,\widetilde{\rho}^*}) \right). \qquad (\star)$$

Remark 6.7. Log-concavity – even in the support-based version – does not hold for ribbon decomposition matrices, as several small counterexamples exist. Furthermore, small counterexamples also arise when using two distinct Temperley–Lieb types in the product of immanants.

We believe that it may be possible to prove Conjecture 6.6 using shuffle tableaux combinatorics paired with the LPP inequality and Littlewood–Richardson semigroup property alone. If this is true, then the constructions and intermediate results from this work may prove useful. The following lemma exemplifies the use of the LPP inequality and Littlewood–Richardson semigroup property in this way:

Lemma 6.8. *In $(\star)$, the minimal partition in the LHS with respect to the dominance order appears in the RHS.*

The following is a sketch of the proof of Lemma 6.8.

Proof (sketch). Let w be the 321-avoiding permutation in S_ℓ with $\tau = t_w$. It suffices to prove that

$$\operatorname{sort}(\lambda_i^* - \mu_{w(i)}^*, \nu_i^* - \rho_{w(i)}^*) \in \operatorname{Supp}_s \left(s_{\operatorname{sort}(\widetilde{\lambda}_i^* - \widetilde{\mu}_{w(i)}^*)} s_{\operatorname{sort}(\widetilde{\nu}_i^* - \widetilde{\rho}_{w(i)}^*)} \right).$$

This follows from an elementary application of the LPP inequality [Spe26, Main Theorem], the Littlewood–Richardson semigroup property and the Pieri identity. ∎

The natural next step would be to prove an analogous result for the maximal partitions. Note that the maximal partition is a function of the canonical minimal strings, and so proving such a result would heavily depend on the geometry of these strings. This could explain the failure of log-concavity when extending to ribbon decomposition matrices.

6.2 Local aspects of support

Having proven the existence of unique maximal and minimal elements in the Schur support, a natural research direction would be to study the support locally at arbitrary elements. Our local framework already supports progress towards this analysis. One local conjecture regarding the support is the following.

Conjecture 6.9. *Let λ/μ be a skew shape of length ℓ, R be an infinite ribbon compatible with λ/μ, and τ be a basis element of $\mathrm{TL}_\ell(2)$ with $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$. For all non-minimal $\nu \in \mathrm{Supp}_s(\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}))$, there exist integers $i < j$ such that $\nu - e_i + e_j \in \mathrm{Supp}_s(\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}))$.*

Our intermediate work proves this conjecture for all partitions ν in the support which are the content of some tableaux T with Temperley–Lieb pattern $\mathbf{S}$ such that there is a square $\square \in T$ with entry x and $x+1 \in \mathsf{Next}(\mathbf{S}, \mathsf{w}(\square))$. Specifically, one may observe that the construction in the proof of Lemma 5.16 changes the content from ν to $\nu - e_i + e_j$, where i is the initial entry of the first increased square and j is the altered entry of the last increased square. For the remaining partitions ν, one may be inclined to directly use the construction in the proof of Proposition 5.18, which takes a Yamanouchi tableau of Temperley–Lieb pattern $\mathbf{S}$ and returns a Yamanouchi tableau of pattern $\varphi(\mathbf{S})$. However, it is generally not true that the resulting content takes the form $\nu - e_i + e_j$ for some $i < j$; therefore, a working construction for these ν requires further work.

6.3 Characterization of support

Across the works of Klyachko [Kly98] and Knutson–Tao [KT99], it is shown that the Schur supports of skew Schur functions can be characterized by a set of linear inequalities known as *Horn's inequalities*. We conjecture that this result extends to Temperley–Lieb immanants of ribbon decomposition matrices.

Conjecture 6.10. *Let λ/μ be a skew shape of length ℓ, R be an infinite ribbon compatible with λ/μ, and τ be a basis element of $\mathrm{TL}_\ell(2)$. Then there exist integers $c(I)$, indexed by subsets $I \subseteq [\ell]$, such that*

$$\mathrm{Supp}_s\left(\mathrm{Imm}_\tau(A_{\lambda/\mu,R})\right) = \left\{\nu \vdash |\lambda/\mu| : \sum_{i\in I} \nu_i \leq c(I)\right\}.$$

There are several localized conjectures which are special cases of Conjecture 6.10. However, we only present one special case, which instead describes necessary and sufficient conditions for when the Schur support for one Temperley–Lieb type is contained in that for another.

Conjecture 6.11. *Let λ/μ be a skew shape of length ℓ, R be an infinite ribbon compatible with λ/μ, and τ, τ' be basis elements of $\mathrm{TL}_\ell(2)$. Then $\mathrm{Supp}_s\left(\mathrm{Imm}_\tau(A_{\lambda/\mu,R})\right) \subseteq \mathrm{Supp}_s\left(\mathrm{Imm}_{\tau'}(A_{\lambda/\mu,R})\right)$ if and only if*

$$\max_{\nu\in\mathrm{Supp}_s\left(\mathrm{Imm}_\tau(A_{\lambda/\mu,R})\right)} \sum_{i\in I} \nu_i \leq \max_{\nu'\in\mathrm{Supp}_s\left(\mathrm{Imm}_{\tau'}(A_{\lambda/\mu,R})\right)} \sum_{i\in I} \nu'_i$$

for each subset $I \subseteq [\ell]$.

One possible approach for using shuffle tableaux to explore conjectures like Conjecture 6.11 is to make local changes to a given Temperley–Lieb pattern which alter the type, and subsequently construct a tableau with the new pattern. A basic example of this approach is the proof of the following result.

Theorem 6.12. *Let λ/μ be a skew shape of length ℓ and τ, τ' be basis elements of $\mathrm{TL}_\ell(2)$ with $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$, $\mathrm{Imm}_{\tau'}^{\mathrm{TL}}(A_{\lambda/\mu,R}) \neq 0$, and $\tau' = \tau \cdot t_i$. Then*

$$\min(\mathrm{Supp}_s(\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\lambda^*,\mu^*}))) <_D \min(\mathrm{Supp}_s(\mathrm{Imm}_{\tau'}(A_{\lambda^*,\mu^*}))).$$

Proof. Note that there exists a square $\square$ such that the orientation of $\square$ in $\mathbf{S}_{\max}(\tau)$ is vertical, and the Temperley–Lieb pattern $\mathbf{S}$ obtained from toggling the orientation of $\square$ has type τ'. Since toggling the orientation of $\square$ does not introduce any (UR) sub-patterns, the filling $T_{\max}(\mathbf{S}, \mathsf{w}(\square_0))$ is a Yamanouchi tableau with Temperley–Lieb pattern $\mathbf{S}$, by Corollary 5.15. Thus, by Lemma 5.16, $T_{\max}(\mathbf{S}, \mathsf{w}(\square_0))$ is elementwise greater than T. By Theorem 1.3 for Temperley–Lieb type τ', $T_{\max}(\mathbf{S}, \mathsf{w}(\square_0))$ has greater content than $T_{\max}[\tau']$, completing the proof. ∎

References

[ACM15] Olga Azenhas, Alessandro Conflitti, and Ricardo Mamede. "Multiplicity-free skew Schur functions with full interval support". In: *Séminaire Lotharingien de Combinatoire* 75 (2015), Art. B75a, 34. URL: https://www.mat.univie.ac.at/~slc/wpapers/s75azconfma.pdf.

[Aze99] Olga Azenhas. "The admissible interval for the invariant factors of a product of matrices". In: *Linear and Multilinear Algebra* 46 (1999), pp. 51–99.

[Cas+23] Federico Castillo et al. "Double Schubert polynomials do have saturated Newton polytopes". In: *Acta Mathematica* 230.1 (2023), pp. 1–61. DOI: 10.4310/ACTA.2023.v230.n1.a1.

[CQ25] Tracy Chin and Daniel Qin. "Symmetric Lorentzian Polynomials". In: *arXiv e-prints* (2025). eprint: 2510.07819v1. URL: https://arxiv.org/abs/2510.07819v1.

[DP07] Galyna Dobrovolska and Pavlo Pylyavskyy. "On products of sln characters and support containment". In: *Journal of Algebra* 316.2 (2007). Computational Algebra, pp. 706–714. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2006.10.033. URL: https://www.sciencedirect.com/science/article/pii/S0021869306006958.

[GHS17] Marisa Gaetz, Will Hardt, and Shruthi Sridhar. "Support equalities among ribbon Schur functions". In: *arXiv preprint arXiv:1709.03011* (2017).

[Hai93] Mark Haiman. "Hecke algebra characters and immanant conjectures". In: *Journal of the American Mathematical Society* 6.3 (1993), pp. 569–595.

[Kly98] Alexander A. Klyachko. "Stable bundles, representation theory and Hermitian operators". In: *Selecta Mathematica, New Series* 4.3 (1998), pp. 419–445. DOI: 10.1007/s000290050037.

[KT99] Allen Knutson and Terence Tao. "The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture". In: *Journal of the American Mathematical Society* 12.4 (1999), pp. 1055–1090. DOI: 10.1090/S0894-0347-99-00299-4.

[LN26] Tuong Le and Son Nguyen. *Skew Hives, Skew Skeps, Skew Schur Log-Concavity*. 2026.

[LP88] Alain Lascoux and Piotr Pragacz. "Ribbon Schur functions". In: *European J. Combin.* 9.6 (1988), pp. 561–574. ISSN: 0195-6698,1095-9971. DOI: 10.1016/S0195-6698(88)80053-2. URL: https://doi.org/10.1016/S0195-6698(88)80053-2.

[LPP07] Thomas Lam, Alexander Postnikov, and Pavlo Pylyavskyy. "Schur positivity and Schur log-concavity". In: *Amer. J. Math.* 129.6 (2007), pp. 1611–1622. ISSN: 0002-9327,1080-6377. DOI: 10.1353/ajm.2007.0045. URL: https://doi.org/10.1353/ajm.2007.0045.

[Lus90] George Lusztig. "Canonical bases arising from quantized enveloping algebras". In: *Journal of the American Mathematical Society* 3.2 (1990), pp. 447–498.

[NNW26] Chau Nguyen, Son Nguyen, and Dora Woodruff. "Shuffle Tableaux, Littlewood–Richardson Coefficients, and Schur Log-Concavity". In: *Algebraic Combinatorics* 9.4 (2026), pp. 1119–1137.

[NP25] Son Nguyen and Pavlo Pylyavskyy. "Temperley–Lieb Crystals". In: *International Mathematics Research Notices* 2025.7 (Apr. 2025), rnaf080. DOI: 10.1093/imrn/rnaf080. URL: https://doi.org/10.1093/imrn/rnaf080.

[NP26] Son Nguyen and Pavlo Pylyavskyy. "Temperley-Lieb Immanants of Ribbon Decomposition Matrices". In: (2026). arXiv: 2605.12880 [math.CO]. URL: https://arxiv.org/abs/2605.12880.

[RS05] Brendon Rhoades and Mark Skandera. "Temperley-lieb immanants". In: *Annals of Combinatorics* 9.4 (2005), pp. 451–494.

[RS06] Brendon Rhoades and Mark Skandera. "Kazhdan–Lusztig immanants and products of matrix minors". In: *Journal of Algebra* 304.2 (2006), pp. 793–811.

[Spe26] David E Speyer. "L-log-concavity and a proof of the conjecture of Lam, Postnikov and Pylyavskyy". In: *arXiv preprint arXiv:2601.05007* (2026).

[Sta84] Richard P. Stanley. "On the number of reduced decompositions of elements of Coxeter groups". In: *European Journal of Combinatorics* 5.4 (1984), pp. 359–372.

[Yev26] Feodor Yevtushenko. "Temperley–Lieb Immanants of Special Jacobi–Trudi Matrices Have Saturated Newton Polytopes". In: (2026).