The St. Petersburg Paradox: Expected Utility and Asymptotic Analysis

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Abstract

The St. Petersburg paradox describes a game with infinite expected value that few would pay much to play. Daniel Bernoulli's 1738 resolution, using a logarithmic utility function, remains the classical starting point for understanding this puzzle. This paper develops a rigorous asymptotic theory of the certainty-equivalent gain (CEG) $D(M)$ as a function of a player's initial wealth M. Using Mellin-transform techniques from analytic combinatorics and complex analysis, we establish a sharp asymptotic expansion $D(M) = 2^{-1}\log_2 M + C_{\mathrm{SP}} + \Delta(\log_2 M) + O\big((\log_2 M)^2/M\big)$ as $M \to \infty$, showing that its leading behavior is $2^{-1}\log_2 M$, with a constant correction $C_{\mathrm{SP}} = 1/(2\ln 2)+1/4$, an extremely small periodic fluctuation and a vanishing remainder term. Thus, the CEG grows primarily at a logarithmic rather than a linear rate with respect to initial wealth. The periodic correction takes the fixed nonzero value $\Delta(0)$ when M is an exact power of 2. A consequence is that $\lim_{M\to\infty}[D(M) - 2^{-1}\log_2 M]$ does not exist in general, though the offset converges to $C_{\mathrm{dyad}} = C_{\mathrm{SP}} + \Delta(0)$ along powers of two. High-precision computations illustrate the expansion and its convergence behavior. We further connect Bernoulli's framework to the Kelly criterion, establishing a unique wealth threshold, determined numerically, above which a log-utility maximizer prefers a risky, high-upside bet over a smaller guaranteed sum.

Keywords: St. Petersburg paradox, Bernoulli utility, asymptotic expansion, Mellin transform, Kelly criterion

1 Introduction

1.1 Historical Context

On September 9, 1713, the Swiss mathematician Nicolas Bernoulli posed a provocative question to his French colleague Pierre Raymond de Montmort. The question concerned a hypothetical game of chance whose expected value appeared to be infinite, yet common sense suggested that no rational individual would pay a substantial sum to participate. This apparent contradiction, which came to be known as the St. Petersburg paradox [1], challenged the foundations of probability theory and decision-making under uncertainty.

The paradox arises from the following game: A player pays an entry fee to participate. A fair coin is tossed repeatedly until it lands on heads. If the first heads appears on the nth toss, the player receives 2^{n-1} ducats. To illustrate: if heads appears on throw 1, the payout

is 1 ducat. If heads appears on throw 2, the payout is 2 ducats. If heads appears on throw 3, the payout is $2^{3-1} = 4$ ducats. In general, for throw n, the payout is 2^{n-1} ducats.

The expected value of the St. Petersburg paradox game is discussed in many textbooks; for example, see [2, 3, 4]. Specifically, The expected value of this game is calculated as follows:

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n 2^{n-1} = \sum_{n=1}^{\infty} \frac{1}{2} = \infty.$$

Despite this infinite expectation, historical discussions questioned even a modest entrance fee: Cramer's letter mentions 20 ducats [6, p. 33]. Daniel Bernoulli challenged the use of expected monetary value as a valuation rule:

> "*Expected values are computed by multiplying each possible gain by the number of ways in which it can occur... The [standard rule] must, therefore, be discarded.*" [5, 6]

1.2 The Central Question

The disconnect between mathematical expectation and rational behavior constitutes the core of the paradox. Instead of calculating expected value, scientists and probabilists strive to model human psychology using algorithms and equations. The key insight is that value is not linear—a dollar means different things to different people depending on their circumstances. In this paper, we explore Bernoulli's resolution of the paradox and extend his analysis to understand the behavior of the game for players with large initial wealth.

2 Daniel Bernoulli's Utility Theory

2.1 The Principle of Diminishing Marginal Utility

Let us consider a simple thought experiment. Suppose you currently have one dollar in your bank account. If someone offered you 100 dollars, you would be ecstatic; you just multiplied your net worth by 101 times. However, if you already have a billion dollars and someone offered you 100 dollars, you would probably think of it as just a cute gesture. In reality, linearly speaking, you have gained the same amount of money, but the satisfaction is completely different. This observation is the foundation of utility theory.

Daniel Bernoulli defined utility as an economic term encompassing one's personal satisfaction and benefit from wealth. He wrote:

> "*The determination of the value of an item must not be based on its price, but rather on the utility it yields⋯⋯; the utility, however, is dependent on the particular circumstances of the person making the estimate.*" [5, 6]

This statement captures the essence of diminishing marginal utility: the more wealth you already have, the less additional utility each new dollar provides. As an extra note, Gabriel Cramer independently reached a similar conclusion in 1728, arguing that money should be evaluated according to its usefulness [7]. This principle is the key to resolving the paradox.

2.2 Derivation of the Logarithmic Utility Function

Bernoulli formalized this intuition using calculus. Let $x > 0$ be current wealth, and let dx be a tiny gain. The change in utility, dy, is given by

$$dy = b \cdot \frac{dx}{x},$$

where $b > 0$ is a constant. This differential equation encodes Bernoulli's hypothesis that utility gain is inversely proportional to current wealth and directly proportional to the amount gained.

By integrating both sides, we get

$$y = b \ln(x) + C.$$

We normalize utility to zero at the initial wealth $a > 0$, so $y = 0$ when $x = a$. Solving for C, we have

$$0 = b \ln(a) + C$$

which gives

$$C = -b \ln(a).$$

Substituting C into $y = b \ln(x) + C$, we obtain

$$y = b \ln(x) - b \ln(a),$$

which can be simplified as

$$y = b \ln \left(\frac{x}{a} \right). \tag{1}$$

The initial wealth a determines the constant of integration $C = -b \ln a$. Looking back at the equation we just derived, the logarithm also satisfies the property that Bernoulli discussed: an increase in utility is inversely proportionate to the amount of goods already possessed. This logarithmic function will be the foundation of our analysis.

2.3 The Expected Utility of the St. Petersburg Game

Now, instead of calculating expected value, we calculate the expected utility of a gain. For the St. Petersburg game, the gain on throw n is 2^{n-1}. The expected utility is

$$\mathbb{E}[U] = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot b \ln \left(\frac{a + 2^{n-1}}{a} \right). \tag{2}$$

The reason for this formula is straightforward. The probability term $(1/2)^n$ is the probability that the first heads appears on the nth throw. Notice how $a + 2^{n-1}$ is present instead of just x. This is because for each possible outcome n, the prize earned (2^{n-1}) is added to

the initial wealth a to represent the new total wealth. We use the sigma summation notation because we sum up all possible cases, from the game ending after one throw to infinite throws. For fixed $a > 0$,

$$0 \leq \ln\left(1 + \frac{2^{n-1}}{a}\right) \leq \ln(1 + 1/a) + (n-1)\ln 2.$$

The upper bound has a finite sum when weighted by 2^{-n}, so the expected logarithmic utility is finite.

3 Determining the Certainty-Equivalent Gain

3.1 Finding the Certainty Equivalent

Having computed the expected utility of the game, we now express its value as a guaranteed monetary gain. For initial wealth $a > 0$, let $D(a)$ denote the sure amount that provides the same utility as receiving the game payoff X for free. Under logarithmic utility, this *certainty-equivalent gain* (CEG) satisfies

$$\ln\big(a + D(a)\big) = \mathbb{E}[\ln(a + X)].$$

The paradox concerns how much a player would pay to enter the game, so it is useful to relate $D(a)$ to a "fair price" for participation. If this term means the maximum acceptable *ticket price* $p(a)$, it generally differs from $D(a)$ because paying the fee reduces the player's wealth. The corresponding indifference condition is

$$\mathbb{E}[\ln(a - p(a) + X)] = \ln a,$$

where $a - p(a) + X > 0$ for every possible payoff. When $a - p(a) > 0$, the two definitions are linked by $p(a) = D(a - p(a))$. Bernoulli already distinguished the value of holding the game from its purchase price [6, pp. 32–33].

To obtain $D(a)$ explicitly, we expand the expected utility as follows:

$$\mathbb{E}[U] = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot b \ln\left(\frac{a + 2^{n-1}}{a}\right).$$

This can be rewritten as

$$\mathbb{E}[U] = b \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\ln(a + 2^{n-1}) - \ln(a)\right].$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, the above equation can be further simplified as

$$\mathbb{E}[U] = b \left[\sum_{n=1}^{\infty} \frac{1}{2^n} \ln(a + 2^{n-1}) - \ln(a)\right].$$

Using the identities $c\ln(x) = \ln(x^c)$ and $\sum \ln(x_n) = \ln(\prod x_n)$, we have

$$\mathbb{E}[U] = b\ln\left[\prod_{n=1}^{\infty}(a+2^{n-1})^{1/2^n}\cdot a^{-1}\right].$$

Writing $D = D(a)$, the utility associated with wealth $a + D$ is

$$U(a+D) = b\ln\left(\frac{a+D}{a}\right).$$

Equating this utility to the expected utility of the gamble gives

$$b\ln\left(\frac{a+D}{a}\right) = b\ln\left[\prod_{n=1}^{\infty}(a+2^{n-1})^{1/2^n}\cdot a^{-1}\right].$$

Canceling b and exponentiating both sides, we obtain

$$\frac{a+D}{a} = \prod_{n=1}^{\infty}(a+2^{n-1})^{1/2^n}\cdot a^{-1}.$$

Multiplying both sides by a yields

$$a+D = \prod_{n=1}^{\infty}(a+2^{n-1})^{1/2^n}.$$

Therefore,

$$D(a) = \prod_{n=1}^{\infty}(a+2^{n-1})^{1/2^n} - a. \tag{3}$$

From now on, we write M for initial wealth. Equation (3) is exact, although numerical evaluation requires truncating the infinite sum or product. As $M \to 0^+$, dominated convergence gives $\mathbb{E}[\ln(M+X)] \to \mathbb{E}[\ln X] = \ln 2$, and hence $D(M) \to 2$ ducats exactly. For $M = 100$ and $M = 1000$ ducats, respectively, we obtain $D(100) \approx 4.38937$ and $D(1000) \approx 5.97225$ ducats, consistent with Bernoulli's rounded values [6, p. 32].

These examples illustrate how the CEG depends on initial wealth. Section 4 shows that its growth is logarithmic as $M \to \infty$. Under logarithmic utility, every positive initial wealth gives a finite valuation, even though the game's expected monetary payoff is infinite.

4 Asymptotic Analysis for Large Wealth

4.1 Motivation and Setup

We now investigate how the CEG $D(M)$ behaves as the initial wealth M grows very large. The aim is to determine whether the gain grows proportionally with wealth or more slowly, and to describe this dependence through an asymptotic expansion.

We begin with the CEG equation

$$D(M) = \prod_{n=1}^{\infty} (M + 2^{n-1})^{1/2^n} - M. \tag{4}$$

Taking the logarithm of both sides after rearranging provides a convenient way to simplify the infinite product, as the logarithm converts the product into a sum. In particular,

$$\ln\left(1 + \frac{D}{M}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(1 + \frac{2^{n-1}}{M}\right). \tag{5}$$

Define

$$S(M) = \sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(1 + \frac{2^{n-1}}{M}\right). \tag{6}$$

Then equation (5) becomes

$$\ln\left(1 + \frac{D}{M}\right) = S(M). \tag{7}$$

and thus

$$D(M) = M\left[e^{S(M)} - 1\right]. \tag{8}$$

Our goal is to determine the asymptotic behavior of $S(M)$ as $M \to \infty$, which will then give us the behavior of $D(M)$. Throughout the paper, we use the standard asymptotic notation; see, for example, Romik [8, Section 0.2]. Specifically, $f(M) = O(g(M))$ means that there exist constants $C > 0$ and $M_0 > 0$ such that $|f(M)| \leq C|g(M)|$ for all $M \geq M_0$. Similarly, $f(M) = o(g(M))$ means that $\lim_{M\to\infty} f(M)/g(M) = 0$.

4.2 Main Results

We first present our asymptotic result for the CEG, which provides a refined characterization of its behavior as initial wealth grows. In particular, the CEG exhibits logarithmic growth, together with a constant correction, an extremely small periodic fluctuation, and a vanishing remainder term.

Theorem 1. *For the St. Petersburg game with initial wealth $M > 0$, the CEG $D(M)$ admits the following asymptotic expansion*

$$D(M) = \frac{1}{2}\log_2 M + C_{\mathrm{SP}} + \Delta(\log_2 M) + O\left(\frac{(\log_2 M)^2}{M}\right)$$

as $M \to \infty$, where

$$C_{\mathrm{SP}} = \frac{1}{2\ln 2} + \frac{1}{4}, \tag{9}$$

and $\Delta(u)$ is a periodic fluctuation term given by

$$\Delta(u) = \frac{1}{2\ln 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} f^*(s_k)e^{-2\pi iku},$$

with $f^*(s) = \pi/[s\sin(\pi s)]$ and $s_k = -1 - 2\pi ik/(\ln 2)$. Moreover, $\Delta(u)$ is uniformly bounded in u by

$$|\Delta(u)| \le \frac{e^{-2\pi^2/(\ln 2)}}{[1-e^{-4\pi^2/(\ln 2)}][1-e^{-2\pi^2/(\ln 2)}]} \approx 4.3\times 10^{-13}.$$

In particular, $\Delta(0) > 0$; the Fourier series is absolutely and uniformly convergent, and Δ has mean zero.

The proof of Theorem 1 is provided in the Appendix. The theorem characterizes the behavior of the CEG $D(M)$ of the St. Petersburg game when the initial wealth M becomes large. It shows that $D(M)$ grows primarily at the logarithmic rate $2^{-1}\log_2 M$, rather than linearly in M. More precisely, the CEG consists of four components: a dominant logarithmic term $2^{-1}\log_2 M$, a constant C_{SP}, a periodic fluctuation $\Delta(\log_2 M)$, and a remainder of order $O((\log_2 M)^2/M)$.

The constant $C_{\mathrm{SP}} = 1/(2\ln 2) + 1/4 \approx 0.97134752044448170368$ provides the non-oscillatory correction to the leading logarithmic term. This constant captures a persistent offset that does not vanish as $M \to \infty$. It therefore forms an essential part of the limiting asymptotic structure rather than being absorbed into the error term.

A further refinement to the asymptotic CEG is provided by the term $\Delta(\log_2 M)$, which describes a periodic fluctuation around the logarithmic trend. Since the function $\Delta(u)$ has period 1, we have $\log_2(2M) = \log_2 M + 1$ and then $\Delta(\log_2 M)$ is unchanged when M is multiplied by 2. The *phase* $u = \log_2 M - \lfloor \log_2 M \rfloor \in [0,1)$ is the fractional part of $\log_2 M$. Writing $M = 2^{m+u}$ with $m \in \mathbb{Z}$ therefore gives $\Delta(\log_2 M) = \Delta(u)$. Thus, although the overall CEG increases logarithmically with wealth, it exhibits a persistent oscillatory component that reflects the dyadic structure of the St. Petersburg game, whose possible payoffs are powers of 2.

Our theorem also shows that the periodic fluctuation $\Delta(u)$ is extremely small. Specifically, the periodic fluctuation $\Delta(u)$ is uniformly bounded in absolute value by the positive constant

$$\frac{e^{-2\pi^2/(\ln 2)}}{[1-e^{-4\pi^2/(\ln 2)}]\,[1-e^{-2\pi^2/(\ln 2)}]} \approx 4.3\times 10^{-13}.$$

Therefore, although $\Delta(\log_2 M)$ is a genuine component of the asymptotic expansion, its magnitude is extraordinarily small. For sufficiently large M, the logarithmic term and C_{SP} therefore give an accurate approximation; at finite M, the error also includes the remainder $O((\log_2 M)^2/M)$. The periodic term is nevertheless mathematically significant because it records the discrete, power-of-two structure of the game that remains visible in the asymptotic expansion.

The final term, $O((\log_2 M)^2/M)$, represents the remainder in the asymptotic approximation. More precisely, its magnitude is bounded by a constant multiple of $(\log_2 M)^2/M$ for all sufficiently large M. Since $(\log_2 M)^2/M \to 0$ as $M \to \infty$, the remainder becomes negligible for sufficiently large wealth.

Finally, when $\log_2 M$ is an integer, the periodic correction takes the fixed value $\Delta(0) > 0$. Define

$$C_{\mathrm{dyad}} = C_{\mathrm{SP}} + \Delta(0) = 0.9713475204449054017858\ldots,$$

where $\Delta(0) \approx 4.2369810587331 \times 10^{-13}$. Thus C_{SP} is the mean offset, while C_{dyad} is the offset along powers of two. This observation yields the following corollary.

Corollary 1. *For the St. Petersburg game with initial wealth $M = 2^m$ with $m \in \mathbb{N}$, the CEG $D(M)$ admits the following asymptotic expansion*

$$D(M) = \frac{1}{2}\log_2 M + C_{\mathrm{dyad}} + O\left(\frac{(\log_2 M)^2}{M}\right)$$

as $M \to \infty$, where $C_{\mathrm{dyad}} = C_{\mathrm{SP}} + \Delta(0)$ and C_{SP} is given in (9).

The proof of Corollary 1 is provided in the Appendix. By Lemma 1, the periodic fluctuation $\Delta(u)$ is nonconstant. There therefore exists $u_0 \in \mathbb{R}$ such that $\Delta(u_0) \neq \Delta(0)$. Combining this observation with Theorem 1 and Corollary 1, we also have the following corollary.

Corollary 2. $\lim_{M\to\infty} [D(M) - 2^{-1}\log_2 M]$ *does not exist.*

The proof of Corollary 2 is given in the Appendix.

5 Numerical Investigation

Using the definition of $D(M)$ in (4), we can compute $D(M)$ numerically for various values of M. Recall that

$$D(M) = M\left[e^{S(M)} - 1\right].$$

Thus, evaluating $D(M)$ reduces to first computing $S(M)$. By the definition of $S(M)$ in (6), we have

$$S(M) = \sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(1 + \frac{2^{n-1}}{M}\right).$$

For numerical computation, we approximate the infinite series by truncating it after N_T terms, i.e.,

$$S(M) \approx \sum_{n=1}^{N_T} \frac{1}{2^n} \ln\left(1 + \frac{2^{n-1}}{M}\right).$$

Substituting this truncated sum into $D(M) = M\left[e^{S(M)} - 1\right]$ yields a numerical approximation of $D(M)$. We take $N_T = 200$ for the numerical computations in this paper. We use 160-digit arithmetic, with `log1p` and `expm1` for stable evaluation of $\ln(1+z)$ and $e^z - 1$. The values were checked using 200-digit arithmetic and an 800-term reference sum. For $M \geq 1$, put $S_N = \sum_{n=1}^{N} 2^{-n} \ln(1 + 2^{n-1}/M)$ and $D_N = M(e^{S_N} - 1)$. Since $1 + 2^{n-1}/M \leq 2^n$,

$$0 \leq S - S_N \leq b_N := \frac{(N+2)\ln 2}{2^N}, \qquad 0 \leq D - D_N \leq (M + D_N)(e^{b_N} - 1).$$

At $N = 200$ and $M = 10^{40}$, the latter bound is less than 8.72×10^{-19}. This bounds truncation; the higher-precision rerun separately checks numerical rounding.

We consider two sequences of values for M:

$$M \in \{2^1, 2^3, 2^5, \ldots, 2^{59}\} \qquad \text{and} \qquad M \in \{10^1, 10^2, 10^3, \ldots, 10^{40}\}.$$

The values in the first sequence are powers of 2, whereas the values in the second sequence are not of the form 2^m for any integer m. The corresponding numerical results are summarized in Table 1 and Figure 1, respectively.

Table 1 reports the values of $D(M)$ and the difference $D(M) - 2^{-1}\log_2 M$ for various values of $M = 2^m$. Two patterns are evident in this special case. First, $D(M)$ increases with M, while the difference $D(M) - 2^{-1}\log_2 M$ decreases and gradually approaches the constant C_{dyad}. At the displayed precision, C_{dyad} and C_{SP} cannot be distinguished. Second, the rate of decrease becomes considerably slower as M increases, consistent with the convergence rate $O((\log_2 M)^2/M)$ established in Corollary 1.

Figure 1(a) illustrates the behavior of $D(M) - 2^{-1}\log_2 M - C_{\text{SP}}$ for $M = 10^j$, $j = 1, \ldots, 40$, which, in general, are not powers of 2. To accommodate the wide range of values of M, the figure plots $D(M) - 2^{-1}\log_2 M - C_{\text{SP}}$ against $\log_{10} M$. It can be seen that the curve starts at approximately 0.41 when $M = 10$, decreases to about 0.0961 at $M = 100$, and falls to approximately 0.018 at $M = 1000$. After $M = 10^5$, the curve is visually indistinguishable from zero and remains so up to $M \sim 10^{40}$. This behavior is consistent with Theorem 1: the vanishing remainder decreases, while the periodic component persists below the visible scale of panel (a). Panel (b) shows $10^{13}\Delta(u)$ for $u \in [0, 1]$, using four conjugate pairs of Fourier terms. The endpoint $u = 1$ repeats $u = 0$, displaying one full period. The coefficient estimate in Lemma 1 bounds the omitted tail in $\Delta(u)$ by $q^5/[(1-q^2)(1-q)] < 2 \times 10^{-62}$, where $q = e^{-2\pi^2/\ln 2}$.

Table 1: Numerical Values of $D(M)$ and $D(M) - 2^{-1}\log_2 M$ for $M \in \{2^1, 2^3, 2^5, \ldots, 2^{59}\}$.

M	$D(M)$	$D(M) - 2^{-1}\log_2 M$	M	$D(M)$	$D(M) - 2^{-1}\log_2 M$
2	2.416374	1.9163738	2^{31}	16.471348	0.9713476
2^3	2.937113	1.4371132	2^{33}	17.471348	0.9713475
2^5	3.675279	1.1752787	2^{35}	18.471348	0.9713475
2^7	4.552381	1.0523814	2^{37}	19.471348	0.9713475
2^9	5.501179	1.0011793	2^{39}	20.471348	0.9713475
2^{11}	6.481705	0.9817049	2^{41}	21.471348	0.9713475
2^{13}	7.474786	0.9747861	2^{43}	22.471348	0.9713475
2^{15}	8.472450	0.9724503	2^{45}	23.471348	0.9713475
2^{17}	9.471692	0.9716916	2^{47}	24.471348	0.9713475
2^{19}	10.471453	0.9714526	2^{49}	25.471348	0.9713475
2^{21}	11.471379	0.9713790	2^{51}	26.471348	0.9713475
2^{23}	12.471357	0.9713568	2^{53}	27.471348	0.9713475
2^{25}	13.471350	0.9713502	2^{55}	28.471348	0.9713475
2^{27}	14.471348	0.9713483	2^{57}	29.471348	0.9713475
2^{29}	15.471348	0.9713477	2^{59}	30.471348	0.9713475

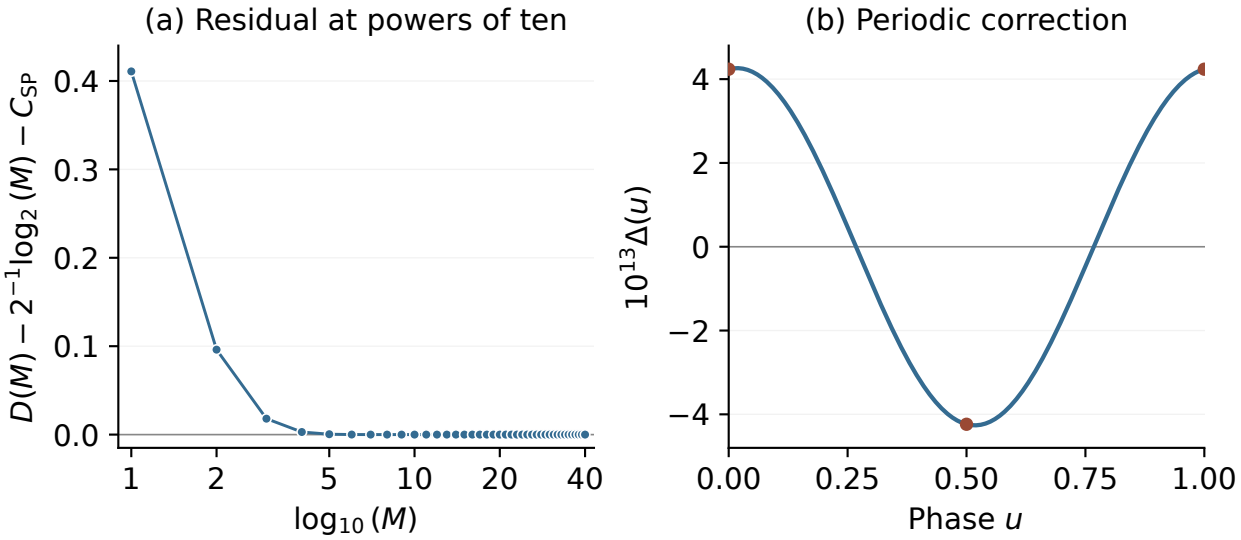


Figure 1: (a) $D(M) - 2^{-1}\log_2 M - C_{\mathrm{SP}}$ against $\log_{10} M$, with a logarithmic horizontal axis. (b) The periodic correction as a function of the phase u (the fractional part of $\log_2 M$), multiplied by 10^{13} for visibility.

6 Connections to the Kelly Criterion

6.1 The Kelly Framework

The Kelly criterion, developed by John L. Kelly Jr. in 1956, provides a framework for determining the optimal fraction of wealth to wager in a favorable gamble [9]. For a gamble with net returns r_i per unit staked, occurring with probabilities p_i, let $W > 0$ denote the initial wealth and f the fraction wagered. The random final wealth W_{final} equals $W + fWr_i = W(1 + fr_i)$ when outcome i occurs. The Kelly criterion chooses f to maximize the expected logarithm of final wealth

$$\mathbb{E}[\ln(W_{\text{final}})] = \sum_i p_i \ln\big(W(1 + fr_i)\big).$$

Without borrowing, take $0 \le f \le 1$ and require $1 + fr_i > 0$ for every outcome with positive probability.

The connection to the St. Petersburg paradox is the shared expected-logarithmic-wealth objective. Bernoulli uses it to value a random payoff, while the Kelly criterion uses it to select a wagering fraction.

6.2 Decision Problems

Consider the following decision problems. These compare fixed payoffs under the same logarithmic objective; they do not optimize a wagering fraction.

Problem 1: Would you prefer 1 million dollars guaranteed, or a 10% chance of receiving 100 million dollars (and nothing otherwise)?

Let initial wealth be M. The utility of the guaranteed option is

$$U_1 = \ln(M + 10^6).$$

The utility of the risky option is

$$U_2 = \frac{1}{10}\ln(M + 10^8) + \frac{9}{10}\ln(M).$$

Under logarithmic utility, we choose the option with higher expected logarithm. For a player with positive initial wealth tending to zero ($M \to 0^+$), we have

$$\begin{aligned} U_1 &\longrightarrow \ln(10^6) \approx 13.82, \\ U_2 &= \frac{1}{10}\ln(M + 10^8) + \frac{9}{10}\ln M \longrightarrow -\infty. \end{aligned}$$

The guaranteed option is clearly superior.

Problem 2: Would you prefer 1 billion dollars guaranteed, or a 10% chance of receiving 100 billion dollars (and nothing otherwise)?

For a player with positive initial wealth tending to zero ($M \to 0^+$), we have

$$\begin{aligned} U_1 &\longrightarrow \ln(10^9) \approx 20.72, \\ U_2 &= \frac{1}{10}\ln(M + 10^{11}) + \frac{9}{10}\ln M \longrightarrow -\infty. \end{aligned}$$

Again, the guaranteed option is superior. The logarithm compresses the values so much that even a 10% chance at 100 billion dollars is not enough to beat a guaranteed 1 billion dollars for a player with negligible initial wealth.

6.3 The Threshold Wealth

For Problem 2, the threshold wealth at which the player is indifferent satisfies

$$\ln(M + 10^9) = \frac{1}{10}\ln(M + 10^{11}) + \frac{9}{10}\ln(M).$$

To show that the threshold is unique, put $x = M/10^9 > 0$ and write the risky option's expected utility minus that of the guaranteed option as

$$g(x) = \frac{1}{10}\ln(1 + 100/x) - \ln(1 + 1/x),$$
$$g'(x) = \frac{9(10 - x)}{x(x + 1)(x + 100)}.$$

Thus g increases on $(0, 10)$ and decreases on $(10, \infty)$. Since $g(x) \to -\infty$ as $x \to 0^+$ and $g(x) \to 0$ as $x \to \infty$, the decreasing branch is positive and there is exactly one zero in $(0, 10)$.

Solving for this zero numerically yields

$$M_* \approx 2.1099859129 \times 10^9 \text{ dollars} \approx 2.11 \text{ billion dollars}.$$

Below M_*, the guaranteed option is strictly preferred; above M_*, the risky option is strictly preferred; and at M_*, the player is exactly indifferent. For these fixed payoffs, logarithmic utility therefore favors the risky option only when initial wealth is sufficiently large.

7 Conclusion

This paper revisits one of the oldest problems in the theory of decision-making under uncertainty and provides a precise, quantitative account of how Bernoulli's classical resolution of the St. Petersburg paradox scales with wealth. Starting from Bernoulli's logarithmic utility function, we derived an exact infinite-product formula for the CEG $D(M)$, valid for every finite initial wealth $M > 0$, and used it to confirm the historical benchmark values reported by Bernoulli and later commentators (e.g., $D(M) \to 2$ ducats as $M \to 0^+$, and $D(1000) \approx 6$ ducats).

Our main contribution is that we have established a sharp, four-term asymptotic expansion of $D(M)$ as $M \to \infty$, which includes a dominant logarithmic growth term $2^{-1}\log_2 M$, an explicit non-oscillatory constant $C_{\mathrm{SP}} = 1/(2\ln 2) + 1/4$, a genuinely periodic but numerically negligible fluctuation term $\Delta(\log_2 M)$ reflecting the underlying dyadic structure of the game, and a remainder that vanishes at the explicit rate $O((\log_2 M)^2/M)$. The asymptotic expansion of $D(M)$ holds uniformly for all sufficiently large real values of M. A byproduct of our analysis is that $\lim_{M\to\infty}[D(M) - 2^{-1}\log_2 M]$ fails to exist in general, due to the periodic term Δ. Along $M = 2^m$, the offset converges to $C_{\mathrm{dyad}} = C_{\mathrm{SP}} + \Delta(0)$; more generally, along $M = 2^{m+u}$ it converges to $C_{\mathrm{SP}} + \Delta(u)$.

The connection with the Kelly criterion is the shared logarithmic objective. In the fixed-payoff comparison of Problem 2, we proved that there is a unique threshold and found numerically that $M_* \approx 2.11$ billion dollars. Within this model, the guaranteed option is preferred below the threshold and the risky option above it; this conclusion depends on the specified payoffs and logarithmic utility.

Several directions remain open for future work. One natural extension is to consider utility functions other than the logarithm (e.g., power utilities $u(w) = w^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$ and $\gamma \neq 1$), and to ask whether an analogous asymptotic expansion, including a periodic correction term reflecting the dyadic payoff structure, persists for this broader class. Another is to examine finite-horizon or capped variants of the St. Petersburg game, which are often used to make the paradox more tractable for classroom or experimental study, and to determine how truncation interacts with the asymptotic structure derived here. We leave these interesting extensions for future research.

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During the preparation of this paper, I used OpenAI's ChatGPT (free version; GPT-5.6) as an AI-assisted tool at several stages of the project. Specifically, I used ChatGPT only to assist with writing and refining computer code for numerical investigations, identifying relevant references and sources, and polishing the language and grammar of drafts. I take full responsibility for the mathematical content, interpretations, conclusions, and final text of the paper.

Appendix A: Proofs of Main Results

Definition 1 (Mellin transform [10])**.** *For a locally integrable function f on $(0,\infty)$, its Mellin transform is defined, wherever the integral converges absolutely, by*

$$f^*(s) = \int_0^\infty t^{s-1} f(t)\, dt, \qquad s \in \mathbb{C}.$$

The fundamental strip is the maximal open vertical strip of absolute convergence.

Conversely, the Mellin inversion of a function $f^*(s)$ is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} f^*(s)\, ds,$$

where c lies within the fundamental strip, provided suitable inversion hypotheses hold. Sufficient conditions are that f is continuous, $t^{c-1}f(t)$ is integrable on $(0,\infty)$, and $f^*(c+i\,\cdot)$ is integrable on $\mathbb{R}$.

A.1. Proof of Theorem 1

Recall that

$$D(M) = M\left[e^{S(M)} - 1\right]$$

where $S(M) = \sum_{n=1}^{\infty} 2^{-n} \ln\left(1 + 2^{n-1}/M\right)$. Therefore, to investigate the asymptotic behavior of $D(M)$ as $M \to \infty$, we first study the asymptotic behavior of $S(M)$ as $M \to \infty$.

The main idea is to treat $S(1/x) = \sum_{n=1}^{\infty} 2^{-n} \ln(1 + 2^{n-1}x)$, with $x = 1/M$, as a harmonic sum [11, 12] and analyze it with the Mellin transform [10]. The transform factors into two pieces: the known Mellin transform of the logarithm function, and a geometric series determined by the weights and scaling factors. Mellin inversion then gives a contour-integral representation of S. Shifting the contour to the left picks up three kinds of poles. Analyzing the residues at these poles yields the asymptotic expansion of $S(M)$. The asymptotic behavior of $D(M)$ then follows from the relation $D(M) = M\left[e^{S(M)} - 1\right]$. For readability, we divide the proof into six steps.

Step 1. Mellin Transform of the Harmonic Sum. Let

$$\lambda_n = 2^{-n}, \qquad \mu_n = 2^{n-1}, \qquad f(t) = \ln(1+t), \quad x = 1/M.$$

Then we have $x \to 0^+$ as $M \to \infty$ and

$$S(1/x) = \sum_{n=1}^{\infty} \lambda_n f(\mu_n x)$$

which is a harmonic sum. Using the standard Mellin transform pair for $f(t) = \ln(1+t)$ (see, for example, [13]), we obtain

$$f^*(s) = \int_0^{\infty} t^{s-1} \ln(1+t)\, dt = \frac{\pi}{s \sin(\pi s)} \tag{10}$$

for any complex number s satisfying $-1 < \mathrm{Re}(s) < 0$. Here $\mathrm{Re}(s)$ is the real part of the complex number s. The exchange of sum and integral below is justified by absolute integrability. For $\sigma = \mathrm{Re}\, s \in (-1, 0)$, scaling gives

$$\sum_{n=1}^{\infty} 2^{-n} \int_0^{\infty} x^{\sigma-1} \ln(1 + 2^{n-1}x)\, dx = \frac{f^*(\sigma)}{2 - 2^{-\sigma}} < \infty.$$

Then the Mellin transform of $S(1/x)$ is

$$S^*(s) = \int_0^{\infty} x^{s-1} \left[\sum_{n=1}^{\infty} \lambda_n f(\mu_n x)\right] dx = \sum_{n=1}^{\infty} \lambda_n \int_0^{\infty} x^{s-1} f(\mu_n x)\, dx. \tag{11}$$

Let $t = \mu_n x$. Then $x = t/\mu_n$ and $dx = dt/\mu_n$. So we have

$$\int_0^{\infty} x^{s-1} f(\mu_n x)\, dx = \int_0^{\infty} \mu_n^{-s} t^{s-1} f(t)\, dt = \mu_n^{-s} \int_0^{\infty} t^{s-1} f(t)\, dt = \mu_n^{-s} f^*(s).$$

This, together with (11), yields

$$S^*(s) = \sum_{n=1}^{\infty} \lambda_n \mu_n^{-s} f^*(s) = f^*(s) A(s), \tag{12}$$

where

$$A(s) = \sum_{n=1}^{\infty} \lambda_n \mu_n^{-s} = \sum_{n=1}^{\infty} (2^{-n})(2^{n-1})^{-s} = 2^s \sum_{n=1}^{\infty} 2^{-n(1+s)} = 2^s \frac{2^{-(1+s)}}{1 - 2^{-(1+s)}} = \frac{1}{2 - 2^{-s}}. \tag{13}$$

Here, the geometric series converges when $|2^{-(1+s)}| < 1$, which is equivalent to $\mathrm{Re}(s) > -1$. Combining (10), (11), and (13), we have

$$S^*(s) = \frac{\pi}{s \sin(\pi s)(2 - 2^{-s})}$$

on the fundamental strip $-1 < \mathrm{Re}(s) < 0$. The series for $S(1/x)$ is locally uniformly convergent, because its terms on a compact x-interval are bounded by a summable multiple of $2^{-n}(1+n)$; hence its sum is continuous. The preceding absolute integral estimate gives the required weighted integrability. On $s = c + it$, with $-1 < c < 0$, the denominator satisfies $|2 - 2^{-s}| \geq 2 - 2^{-c} > 0$, while the reciprocal sine factor decays exponentially as $|t| \to \infty$. Thus $S^*(c + i\,\cdot)$ is integrable, and Mellin inversion gives

$$S(1/x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} S^*(s)\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{\pi}{s \sin(\pi s)(2 - 2^{-s})}\, ds \tag{14}$$

where $-1 < c < 0$. We will use this integral representation as the starting point for the asymptotic expansion.

Step 2: Pole Structure of $x^{-s}S^*(s)$. The displayed formula for $S^*(s)$ gives its meromorphic continuation beyond the fundamental strip. For $M \to \infty$, we have $x \to 0^+$ so we shift the contour to the left. According to the Mellin asymptotic principle, the poles to the left of the fundamental strip produce the asymptotic expansion as $x \to 0^+$. Since $x > 0$, we have $x^{-s}S^*(s) = e^{-s \ln x} S^*(s)$ where $e^{-s \ln x}$ is entire and nonvanishing in s. Thus, $x^{-s}S^*(s)$ and $S^*(s)$ have exactly the same singularities and pole orders. Noting that $S^*(s) = f^*(s)A(s)$, the poles of $S^*(s)$ come from two sources: those of $f^*(s)$ and those of $A(s)$.

The poles we encounter from $f^*(s) = \pi/[s \sin(\pi s)]$ are

$$s = -1, -2, -3, \cdots.$$

At each $s = k \in \{-1, -2, -3, \cdots\}$, we have $\sin(\pi s) = \sin[\pi k + \pi(s - k)] \sim (-1)^k \pi (s - k)$ and thus

$$f^*(s) = \frac{(-1)^k}{k} \cdot \frac{1}{s - k} + O(1) \qquad (s \to k).$$

So every negative integer is a simple pole of $f^*(s)$ [14, Ch. 4].

In view of (13), the poles from $A(s)$ are given by $2-2^{-s}=0$. Since $2^{-s}=e^{-s\ln 2}$, this equation is equivalent to $e^{-s\ln 2}=e^{\ln 2}$. Thus, $-s\ln 2=\ln 2+2\pi ik$ with $k\in\mathbb{Z}$ and the poles from $A(s)$ are located at

$$s_k=-1-\frac{2\pi ik}{\ln 2},\qquad k\in\mathbb{Z}.$$

Each s_k is a simple pole of $A(s)$ because

$$\frac{d}{ds}(2-2^{-s})=(\ln 2)2^{-s}, \tag{15}$$

which is nonzero at each s_k.

Since $s=-1$ is a simple pole of both $f^*(s)$ and $A(s)$, it is a double pole of their product $S^*(s)$. All other points $s_k=-1-2\pi ik/(\ln 2)$ with $k\in\mathbb{Z}\backslash\{0\}$ are simple poles of $S^*(s)$. In addition, all other negative integers $s=-2,-3,\cdots$ are also simple poles of $S^*(s)$. Consequently, $s=-1$ is a double pole of $x^{-s}S^*(s)$, while $s_k=-1-2\pi ik/(\ln 2)$ with $k\in\mathbb{Z}\setminus\{0\}$ and $s=-2,-3,\ldots$ are simple poles of $x^{-s}S^*(s)$.

To justify the contour shift, set $d=-2-\delta$, $0<\delta<1$, and use rectangles with horizontal sides at $T_j=(2j+1)\pi/\ln 2$, $j=1,2,\ldots$. For $d\le\sigma\le c$,

$$|2-2^{-(\sigma\pm iT_j)}|=2+2^{-\sigma}\ge 2.$$

The reciprocal sine factor is $O(e^{-\pi T_j})$ uniformly on these sides, $|\sigma\pm iT_j|^{-1}\le T_j^{-1}$, and $x^{-\sigma}$ is bounded there for each fixed $x>0$. The horizontal integrals therefore vanish. Lemma 1 gives absolute convergence of the residue series, and Lemma 2 bounds the new vertical integral. The residue theorem consequently applies; compare [11, Corollary 1 to Theorem 4, p. 23; Example 12, pp. 30–31]. Thus, we have

$$\begin{aligned}S(M)=S(1/x)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}S^*(s)\,ds\\&=\operatorname{Res}(x^{-s}S^*(s),-1)+\sum_{k\in\mathbb{Z}\backslash\{0\}}\operatorname{Res}(x^{-s}S^*(s),s_k)+\operatorname{Res}(x^{-s}S^*(s),-2)+R_2(x)\end{aligned} \tag{16}$$

with $x=1/M$ and

$$R_2(x)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}x^{-s}S^*(s)\,ds \tag{17}$$

where $d=-2-\delta$ and $0<\delta<1$.

Step 3: The Double Pole at $s=-1$. Setting $s=-1+\varepsilon$ in (10) and (13), we have

$$\begin{aligned}f^*(-1+\varepsilon)&=\frac{\pi}{(-1+\varepsilon)\sin(\pi(-1+\varepsilon))}=\frac{\pi}{(1-\varepsilon)\sin(\pi\varepsilon)},\\A(-1+\varepsilon)&=\frac{1}{2-2^{-(-1+\varepsilon)}}=\frac{1}{2}\left(\frac{1}{1-2^{-\varepsilon}}\right)=\frac{1}{2}\left(\frac{1}{1-e^{-\varepsilon\ln 2}}\right).\end{aligned}$$

Using the Laurent expansion of the cosecant function [15, Eq. 1.411(11)],

$$\frac{\pi}{\sin(\pi\varepsilon)} = \frac{1}{\varepsilon} + \frac{\pi^2}{6}\varepsilon + \frac{7\pi^4}{360}\varepsilon^3 + O(\varepsilon^5)$$

and

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \cdots$$

for $|\varepsilon| < 1$, we obtain the following Laurent expansion

$$\begin{aligned} f^*(-1+\varepsilon) =& \frac{1}{\varepsilon} + 1 + \left(1 + \frac{\pi^2}{6}\right)\varepsilon + \left(1 + \frac{\pi^2}{6}\right)\varepsilon^2 + \left(1 + \frac{\pi^2}{6} + \frac{7\pi^4}{360}\right)\varepsilon^3 + O(\varepsilon^4) \\ =& \frac{1}{\varepsilon} + 1 + O(\varepsilon) \end{aligned}$$

as $\varepsilon \to 0$.

Using the standard expansion (see, for example, [15, Eq. 1.213])

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \frac{t^6}{30240} + O(t^8)$$

for $t \to 0$, we have

$$\frac{1}{1-e^{-t}} = \frac{1}{t}\left(\frac{t}{e^t - 1}\right) + 1 = \frac{1}{t} + \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \frac{t^5}{30240} + O(t^7).$$

Taking $t = \varepsilon \ln 2$ gives

$$A(-1+\varepsilon) = \frac{1}{2\ln 2}\varepsilon^{-1} + \frac{1}{4} + O(\varepsilon)$$

as $\varepsilon \to 0$. Thus, we have

$$S^*(-1+\varepsilon) = f^*(-1+\varepsilon)A(-1+\varepsilon) = \frac{1}{2\ln 2}\varepsilon^{-2} + \left(\frac{1}{2\ln 2} + \frac{1}{4}\right)\varepsilon^{-1} + O(1).$$

Since $x^{-\varepsilon} = e^{-\varepsilon \ln x}$ for any $x > 0$, the Taylor expansion

$$e^z = 1 + z + O(z^2), \qquad z \to 0$$

(see, for example, [16]) yields $x^{-\varepsilon} = 1 - \varepsilon \ln x + O(\varepsilon^2)$ as $\varepsilon \to 0$ and therefore

$$x^{-s} = x\, x^{-\varepsilon} = x\left(1 - \varepsilon \ln x + O(\varepsilon^2)\right).$$

Hence, the residue at $s = -1$ is given by the coefficient of ε^{-1} in the Laurent expansion of $x^{-s}S^*(s)$ around $s = -1$, namely,

$$\operatorname{Res}\left(x^{-s}S^*(s), -1\right) = x\left[\frac{-\ln x}{2\ln 2} + \frac{1}{2\ln 2} + \frac{1}{4}\right] = \frac{1}{M}\left[\frac{1}{2}\log_2 M + C_{\mathrm{SP}}\right] \tag{18}$$

where we have used $x = 1/M$ and

$$C_{\mathrm{SP}} = \frac{1}{2\ln 2} + \frac{1}{4}. \tag{19}$$

Step 4: The Simple Poles $s_k = -1 - 2\pi ik/(\ln 2)$ with $k \in \mathbb{Z}\backslash\{0\}$ (Periodic Correction). The residue of $x^{-s}S^*(s)$ at each $s = s_k$ ($k \neq 0$) is

$$\begin{aligned}\mathrm{Res}\left(x^{-s}S^*(s), s_k\right) &= \lim_{s\to s_k}\left[(s-s_k)x^{-s}S^*(s)\right] = \frac{x^{-s_k}\pi}{s_k \sin(\pi s_k)} \lim_{s\to s_k}\left[(s-s_k)\frac{1}{2-2^{-s}}\right] \\ &= x^{-s_k} f^*(s_k)\,\mathrm{Res}\left(A(s), s_k\right). \end{aligned} \tag{20}$$

Recall that each s_k ($k \neq 0$) is a simple pole of $A(s)$. In view of (15) and by the residue formula for a simple pole [16, Sec. 4.1], we have

$$\mathrm{Res}\left(A(s), s_k\right) = \frac{1}{2^{-s_k}\ln 2}. \tag{21}$$

Since $s_k = -1 - 2\pi ik/(\ln 2)$ and $x = 1/M$, we obtain

$$x^{-s_k} = x \cdot x^{2\pi ik/\ln 2} = M^{-1} \cdot e^{-2\pi ik \log_2 M} \quad \text{and} \quad 2^{-s_k} = 2.$$

Therefore, combining this with (20) and (21), the residue of $x^{-s}S^*(s)$ at each $s = s_k$ ($k \neq 0$) can be written as

$$\mathrm{Res}\left(x^{-s}S^*(s), s_k\right) = M^{-1} \cdot e^{-2\pi ik \log_2 M} f^*(s_k) \frac{1}{2\ln 2}.$$

Define

$$\Delta(u) = \frac{1}{2\ln 2} \sum_{k\neq 0} f^*(s_k) e^{-2\pi iku}. \tag{22}$$

Then we have

$$\sum_{k\neq 0} \mathrm{Res}\left(x^{-s}S^*(s), s_k\right) = \sum_{k\neq 0}\left[M^{-1} \cdot e^{-2\pi ik \log_2 M} f^*(s_k)\frac{1}{2\ln 2}\right] = \frac{1}{M}\Delta(\log_2 M). \tag{23}$$

As we have shown in Lemma 1, the function $\Delta(u)$ is real-valued, bounded, and periodic with period 1. Moreover, we have

$$|\Delta(u)| \leq \frac{e^{-2\pi^2/(\ln 2)}}{[1 - e^{-4\pi^2/(\ln 2)}][1 - e^{-2\pi^2/(\ln 2)}]} \approx 4.3 \times 10^{-13}$$

for any real u and $\Delta(0) > 0$.

Step 5: Remainder. The pole at $s = -2$ contributes

$$\mathrm{Res}\big(x^{-s}S^*(s), -2\big) = x^2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{x^2}{4}.$$

Taking $N = 2$ in Lemma 2 gives

$$|R_2(x)| \leq \frac{C_\delta}{2\pi} x^{2+\delta}$$

for any $x > 0$, where $C_\delta > 0$ is a constant independent of x. Let $x = 1/M$, we have

$$|R_2(x)| = O\left(\frac{1}{M^{2+\delta}}\right) = o\left(\frac{1}{M^2}\right). \tag{24}$$

Combining (16), (18), (23), and (24) gives

$$S(M) = \frac{1}{M}\left[\frac{1}{2}\log_2 M + C_{\mathrm{SP}} + \Delta(\log_2 M)\right] + \frac{1}{4M^2} + O\left(\frac{1}{M^{2+\delta}}\right) \tag{25}$$

with $C_{\mathrm{SP}} = 1/(2\ln 2) + 1/4$ as $M \to \infty$.

Step 6: Exponentiating to Recover $D(M)$. Now, we are ready to investigate the asymptotic behavior of $D(M)$ as $M \to \infty$. It follows immediately from (25) that $S(M) = O\left((\log_2 M)/M\right) \to 0$ as $M \to \infty$. Recall that $D(M) = M\left[e^{S(M)} - 1\right]$. Using $e^S - 1 = S + S^2/2 + O(S^3)$ yields

$$D(M) = MS(M) + \frac{MS^2(M)}{2} + O(MS^3(M)). \tag{26}$$

Let $F(M) = 2^{-1}\log_2 M + C_{\mathrm{SP}} + \Delta(\log_2 M)$. Then $F(M) = O(\log_2 M)$ and it follows from (25) that

$$S(M) = \frac{F(M)}{M} + \frac{1}{4M^2} + O\left(\frac{1}{M^{2+\delta}}\right).$$

Thus, we have

$$\begin{aligned}
MS(M) =& F(M) + \frac{1}{4M} + O\left(\frac{1}{M^{1+\delta}}\right), \\
\frac{MS^2(M)}{2} =& \frac{M}{2}\left[\frac{F^2(M)}{M^2} + 2\frac{F(M)}{M}\cdot O\left(\frac{1}{M^2}\right) + O\left(\frac{1}{M^4}\right)\right] = \frac{F^2(M)}{2M} + O\left(\frac{\log_2 M}{M^2}\right), \\
MS^3(M) =& M \cdot O\left(\left[\frac{(\log_2 M)}{M}\right]^3\right) = O\left(\frac{(\log_2 M)^3}{M^2}\right)
\end{aligned}$$

Combining these results with (26) and the definition of $F(M)$, we have

$$D(M) = \frac{1}{2}\log_2 M + C_{\mathrm{SP}} + \Delta(\log_2 M) + O\left(\frac{(\log_2 M)^2}{M}\right)$$

as $M \to \infty$. More precisely, the same estimates give

$$D(M) - F(M) = \frac{F(M)^2/2 + 1/4}{M} + O\left(M^{-1-\delta} + \frac{(\log_2 M)^3}{M^2}\right).$$

Since $F(M)/\log_2 M \to 1/2$, we have $M[D(M) - F(M)]/(\log_2 M)^2 \to 1/8$, so the remainder order is sharp. This completes the proof of Theorem 1.

A.2. Proof of Corollary 1

Recall from Lemma 1 that $\Delta(u+1) = \Delta(u)$ and $\Delta(0) > 0$. Therefore, for any $M = 2^m$ with $m \in \mathbb{N}$, we have $\Delta(\log_2 M) = \Delta(m) = \Delta(0)$. Combining this with the result in Theorem 1 gives

$$D(M) = \frac{1}{2}\log_2 M + C_{\text{dyad}} + O\left(\frac{(\log_2 M)^2}{M}\right).$$

Here $C_{\text{dyad}} = C_{\text{SP}} + \Delta(0)$. This completes the proof of Corollary 1.

A.3. Proof of Corollary 2

First consider the subsequence $M = 2^m$ with $m \in \mathbb{N}$. It follows from the proof of Corollary 1 that $\Delta(\log_2 M) = \Delta(m) = \Delta(0)$, for any $M = 2^m$ and hence

$$D(2^m) - 2^{-1}\log_2(2^m) \longrightarrow C_{\text{SP}} + \Delta(0) = C_{\text{dyad}}.$$

On the other hand, by Lemma 1, the periodic fluctuation $\Delta(u)$ is nonconstant. There exists $u_0 \in \mathbb{R}$ such that $\Delta(u_0) \neq \Delta(0)$. Then consider the sequence $M_m = 2^{m+u_0}$. Since $\log_2 M_m = m + u_0$, the periodicity of Δ yields

$$\Delta(\log_2 M_m) = \Delta(m + u_0) = \Delta(u_0).$$

Consequently,

$$D(M_m) - 2^{-1}\log_2 M_m \longrightarrow C_{\text{SP}} + \Delta(u_0) \neq C_{\text{SP}} + \Delta(0).$$

Thus, $\lim_{M\to\infty}[D(M) - 2^{-1}\log_2 M]$ does not exist. This completes the proof of Corollary 2.

Appendix B: Lemmas and their Proofs

Lemma 1. *The function $\Delta(u)$, defined in (22), is real-valued, bounded, and periodic with period 1. Moreover, $\Delta(u)$ satisfies the following properties:*

a) $\Delta(u)$ is uniformly bounded in u by

$$|\Delta(u)| \leq \frac{e^{-2\pi^2/(\ln 2)}}{[1 - e^{-4\pi^2/(\ln 2)}][1 - e^{-2\pi^2/(\ln 2)}]} \approx 4.3 \times 10^{-13}.$$

b) $\Delta(0) > 0$, and $\int_0^1 \Delta(u)\, du = 0$.

c) $\Delta(u)$ is a nonconstant function of u.

Proof. Recall that

$$\Delta(u) = \frac{1}{2\ln 2} \sum_{k\neq 0} f^*(s_k) e^{-2\pi i k u}.$$

The coefficient estimate below proves absolute and uniform convergence, which justifies all pairings and termwise integrals in this proof. Since $e^{-2\pi i k(u+1)} = e^{-2\pi i k u}$, it follows immediately that $\Delta(u+1) = \Delta(u)$, meaning that $\Delta(u)$ is a 1-periodic function.

Moreover, since $s_k = -1 - 2\pi i k/(\ln 2)$, we have $s_{-k} = \overline{s_k}$. Because π is real and $\sin(\pi\overline{s}) = \overline{\sin(\pi s)}$, we get

$$f^*(\overline{s}) = \frac{\pi}{\overline{s}\sin(\pi\overline{s})} = \frac{\pi}{\overline{s\sin(\pi s)}} = \overline{\frac{\pi}{s\sin(\pi s)}} = \overline{f^*(s)}$$

and thus

$$f^*(s_{-k}) e^{-2\pi i(-k)u} = f^*(\overline{s_k}) e^{-2\pi i(-k)u} = \overline{f^*(s_k)} e^{-2\pi i(-k)u} = \overline{f^*(s_k) e^{-2\pi i k u}},$$

Hence, the terms corresponding to k and $-k$ in the definition of $\Delta(u)$ are complex conjugates. Consequently, $\Delta(u)$ is real-valued and can be written as

$$\Delta(u) = \frac{1}{2\ln 2} \sum_{k=1}^{\infty} f^*(s_k) e^{-2\pi i k u} + \frac{1}{2\ln 2} \sum_{k=1}^{\infty} f^*(s_{-k}) e^{-2\pi i(-k)u} = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \operatorname{Re}\left[f^*(s_k) e^{-2\pi i k u}\right].$$

Therefore, $\Delta(u)$ is a real-valued function.

Next, we will show that $\Delta(u)$ is bounded. For any real number u, we have

$$|\Delta(u)| \leq \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left|\operatorname{Re}\left[f^*(s_k) e^{-2\pi i k u}\right]\right| \leq \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left|f^*(s_k) e^{-2\pi i k u}\right| = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left|f^*(s_k)\right|. \quad (27)$$

For any positive integer k, let $a_k = 2\pi k/(\ln 2) > 0$. Then $a_1 = 2\pi/(\ln 2)$, $s_k = -1 - 2\pi i k/(\ln 2) = -1 - i a_k$, $|s_k| = \sqrt{1 + a_k^2}$ and

$$\sin(\pi s_k) = \sin(-\pi - i\pi a_k) = \sin(-\pi)\cosh(-\pi a_k) + i\cos(-\pi)\sinh(-\pi a_k) = i\sinh(\pi a_k).$$

Thus, we have

$$|f^*(s_k)| = \left|\frac{\pi}{s_k \sin(\pi s_k)}\right| = \frac{\pi}{|s_k| \cdot |\sin(\pi s_k)|} = \frac{\pi}{\sqrt{1 + a_k^2} \cdot \sinh(\pi a_k)}.$$

Since $a_k = k a_1 > 0$ by the definition of a_k and

$$\sinh(\pi a_k) = \frac{e^{\pi k a_1} - e^{-\pi k a_1}}{2} \geq \frac{e^{\pi k a_1}}{2}(1 - e^{-2\pi k a_1}) \geq \frac{e^{\pi k a_1}}{2}(1 - e^{-2\pi a_1}),$$

we have

$$|f^*(s_k)| \leq \frac{2\pi}{k a_1 (1 - e^{-2\pi a_1})} e^{-\pi k a_1} \leq \frac{2\pi}{a_1 (1 - e^{-2\pi a_1})} e^{-\pi k a_1}.$$

This, together with (27) and $a_1 = 2\pi/(\ln 2)$, gives

$$\begin{aligned}|\Delta(u)| \leq & \frac{1}{\ln 2}\sum_{k=1}^{\infty}\frac{2\pi}{a_1(1-e^{-2\pi a_1})}e^{-\pi k a_1} = \frac{2\pi}{a_1(1-e^{-2\pi a_1})\ln 2}\sum_{k=1}^{\infty}e^{-\pi k a_1} \\ = & \frac{2\pi}{a_1(1-e^{-2\pi a_1})\ln 2}\cdot\frac{e^{-\pi a_1}}{1-e^{-\pi a_1}} = \frac{e^{-2\pi^2/(\ln 2)}}{[1-e^{-4\pi^2/(\ln 2)}][1-e^{-2\pi^2/(\ln 2)}]} \\ \approx & 4.3\times 10^{-13}\end{aligned} \tag{28}$$

for any real number u.

The geometric majorant above also proves uniform convergence by the Weierstrass test. Termwise integration over $[0,1]$ is therefore valid. Since each nonzero Fourier mode has integral zero, so Δ has mean zero; that is,

$$\int_0^1 \Delta(u)\,du = 0.$$

Next, we will show $\Delta(0) > 0$. By the definition of $\Delta(u)$, we have

$$\Delta(0) = \frac{1}{2\ln 2}\sum_{k\neq 0} f^*(s_k)e^{-2\pi i k(0)} = \frac{1}{2\ln 2}\sum_{k\neq 0} f^*(s_k).$$

Recall that $s_k = -1 - ika_1$, we have $\sin(\pi s_k) = \sin(-\pi - i\pi k a_1) = i\sinh(\pi k a_1)$ and

$$f^*(s_k) = \frac{\pi}{s_k\sin(\pi s_k)} = \frac{\pi}{(-1-ika_1)i\sinh(\pi k a_1)} = \frac{\pi}{(ka_1 - i)\sinh(\pi k a_1)}.$$

On the other hand, for $-k$, we have $s_{-k} = -1 - i(-k)a_1 = -1 + ika_1$ and thus

$$\begin{aligned}&\sin(\pi s_{-k}) = \sin(-\pi + i\pi k a_1) = -i\sinh(\pi k a_1), \\ f^*(s_{-k}) = & \frac{\pi}{s_{-k}\sin(\pi s_{-k})} = \frac{\pi}{(-1+ika_1)[-i\sinh(\pi k a_1)]} = \frac{\pi}{(ka_1+i)\sinh(\pi k a_1)} = \overline{f^*(s_k)}.\end{aligned}$$

Hence, the terms $f^*(s_{-k})$ and $f^*(s_k)$ are conjugates, and their sum is

$$f^*(s_{-k}) + f^*(s_k) = \frac{2\pi k a_1}{(1+k^2a_1^2)\sinh(\pi k a_1)} > 0.$$

The absolute convergence already proved permits pairing, giving

$$\Delta(0) = \frac{\pi}{\ln 2}\sum_{k=1}^{\infty}\frac{ka_1}{(1+k^2a_1^2)\sinh(\pi k a_1)} > 0.$$

Finally, we will show that $\Delta(u)$ is a nonconstant function of u. This is equivalent to showing that there exists $u_0 \in \mathbb{R}$ such that $\Delta(u_0) \neq \Delta(0)$. Recall that the Fourier representation of $\Delta(u)$ is

$$\Delta(u) = \frac{1}{2\ln 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} f^*(s_k)e^{-2\pi i k u},$$

where

$$f^*(s) = \frac{\pi}{s\sin(\pi s)} \qquad \text{and} \qquad s_k = -1 - \frac{2\pi i k}{\ln 2}.$$

For every $k \neq 0$, we have $s_k \neq 0$, and s_k is not an integer since it has a nonzero imaginary part. Hence

$$\sin(\pi s_k) \neq 0,$$

so that

$$f^*(s_k) = \frac{\pi}{s_k \sin(\pi s_k)} \neq 0.$$

In particular, the Fourier coefficient corresponding to $k = 1$,

$$\frac{1}{2\ln 2} f^*(s_1),$$

is nonzero. By uniqueness of Fourier coefficients (see, for example, [17, Ch. 2]), $\Delta(u)$ cannot be constant. It follows that there exists $u_0 \in \mathbb{R}$ such that $\Delta(u_0) \neq \Delta(0)$. □

Lemma 2. *Let*

$$R_N(x) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} x^{-s} S^*(s)\, ds,$$

where $d = -N - \delta$, $N \geq 1$ is an integer, and $0 < \delta < 1$. Then

$$|R_N(x)| \leq \frac{C_\delta}{2\pi} x^{N+\delta} \tag{29}$$

for any $x > 0$.

Proof. Let $s = d + it$, where $d = -N - \delta$ is fixed. Thus, $ds = idt$ and

$$\begin{aligned} |R_N(x)| &= \left| \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} x^{-s} S^*(s)\, ds \right| = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-d-it} S^*(d+it)\, dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |x^{-d-it} S^*(d+it)|\, dt = \frac{x^{N+\delta}}{2\pi} \int_{-\infty}^{\infty} |S^*(d+it)|\, dt. \end{aligned} \tag{30}$$

The integral $\int_{-\infty}^{\infty} |S^*(d+it)|\, dt$ can be decomposed into two parts

$$\int_{-\infty}^{\infty} |S^*(d+it)|\, dt = \int_{|t|>1} |S^*(d+it)|\, dt + \int_{-1}^{1} |S^*(d+it)|\, dt \triangleq I_1 + I_2 \tag{31}$$

We first consider I_1. Recall that

$$S^*(d+it) = \frac{\pi}{(d+it)\sin(\pi(d+it))(2 - 2^{-d-it})}.$$

An application of the reverse triangle inequality gives

$$|2 - 2^{-d-it}| \geq |2^{-d-it}| - |2| = 2^{-d} - 2 = 2^{N+\delta} - 2 > 0. \tag{32}$$

In addition, $|d+it| = \sqrt{d^2+t^2} > |t|$. Using the identity $|\sin(\sigma+iy)|^2 = \sin^2\sigma + \sinh^2 y$ for real numbers σ and y [18, Ch. 3], we have

$$|\sin(\pi(d+it))|^2 = \sin^2(\pi d) + \sinh^2(\pi t) \geq \sinh^2(\pi t),$$

and thus $|\sin(\pi(d+it))| \geq |\sinh(\pi t)| = \sinh(|\pi t|) = \sinh(\pi|t|)$. For $|t| \geq 1$,

$$\sinh(\pi|t|) = \frac{e^{\pi|t|} - e^{-\pi|t|}}{2} = \frac{e^{\pi|t|}}{2}(1 - e^{-2\pi|t|}) \geq \frac{e^{\pi|t|}}{2}(1 - e^{-2\pi}).$$

Putting everything together yields

$$|S^*(d+it)| = \left|\frac{\pi}{(d+it)\sin(\pi(d+it))(2-2^{-d-it})}\right| < \frac{1}{|t|}\frac{2\pi}{1-e^{-2\pi}}e^{-\pi|t|}\frac{1}{2^{-d}-2} = C_d\frac{e^{-\pi|t|}}{|t|}$$

for $|t| > 1$, where $C_d = 2\pi/[(1-e^{-2\pi})(2^{-d}-2)] > 0$ is a constant depending only on d (and not on t or x). Consequently, we have

$$\begin{aligned} I_1 = \int_{|t|>1} |S^*(d+it)|\,dt &\leq C_d \int_{|t|>1} \frac{e^{-\pi|t|}}{|t|}\,dt \\ &\leq C_d \int_{|t|>1} e^{-\pi|t|}\,dt = 2C_d\int_1^\infty e^{-\pi t}\,dt = \frac{2C_d}{\pi}e^{-\pi}. \end{aligned} \tag{33}$$

Next, we deal with I_2. Recall that $d = -N-\delta$, where $N \geq 1$ is an integer. Using again the identity $|\sin(\sigma+iy)|^2 = \sin^2\sigma + \sinh^2 y$ for real numbers σ and y [18, Ch. 3] gives

$$|\sin(\pi(d+it))|^2 = \sin^2(\pi d) + \sinh^2(\pi t) \geq \sin^2(\pi d),$$

and thus $|\sin(\pi(d+it))| \geq |\sin(\pi d)| = |\sin(-N\pi - \pi\delta)| = \sin(\pi\delta) > 0$. Moreover, $|d+it| = \sqrt{d^2+t^2} \geq |d| = N+\delta \geq 1+\delta > 0$. Therefore, combining these results with (32) yields

$$|S^*(d+it)| = \left|\frac{\pi}{(d+it)\sin(\pi(d+it))(2-2^{-d-it})}\right| \leq \frac{\pi}{(N+\delta)(2^{N+\delta}-2)\sin(\pi\delta)}$$

for every $|t| \leq 1$. Consequently, we have

$$I_2 = \int_{-1}^1 |S^*(d+it)|\,dt \leq \frac{2\pi}{(N+\delta)(2^{N+\delta}-2)\sin(\pi\delta)} \leq \frac{2\pi}{(1+\delta)(2^{1+\delta}-2)\sin(\pi\delta)} \tag{34}$$

by noting that $-d = N+\delta \geq 1+\delta > 0$ and $2^{-d}-2 = 2^{N+\delta}-2 \geq 2^{1+\delta}-2 > 0$. Recall that $C_d = 2\pi/[(1-e^{-2\pi})(2^{-d}-2)] > 0$, we have

$$C_d \leq \frac{2\pi}{(1-e^{-2\pi})(2^{1+\delta}-2)}.$$

Define

$$C_\delta = \frac{4e^{-\pi}}{(1-e^{-2\pi})(2^{1+\delta}-2)} + \frac{2\pi}{(1+\delta)(2^{1+\delta}-2)\sin(\pi\delta)} > 0$$

which is a constant depending only on δ. Combining this with (30), (31), (33), and (34), we have

$$|R_N(x)| \leq \frac{C_\delta}{2\pi}x^{N+\delta}$$

for any $x > 0$. □

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