Counterexamples to Jeffs' Conjecture on Convex Neural Codes

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Abstract

A neural code lists the sets of neurons that fire together, and it is called convex when the firing regions can be taken to be convex open subsets of Euclidean space. In 2025, Ahmed, Crepeau, Flores, Isekenegbe, Perez and Shiu divided the codes with four maximal codewords into 20 classes according to the nerve of those codewords. For the first 15 classes, and for the minimal codes of the sixteenth, they showed that a code is convex when it has neither a local obstruction nor a wheel. The non-minimal L24 codes and the last four classes, L25–L28, remained open, and they recorded Jeffs' conjecture that the same equivalence holds for all codes with at most four maximal codewords. This report settles L28 for minimal codes and disproves the conjecture. A type reduction shows that the minimal L28 codes, on any number of neurons, are controlled by 606 explicit complexes. A cone lemma reduces 353 of these to classes that are already settled. The other 253 fall into three groups: 115 convex codes, 80 codes with sprockets, and 58 codes that are non-convex, have no local obstruction, and have a realization without a wheel. Two new arguments establish the non-convexity of the 58. An absorption lemma for three convex open sets handles 18 of them, and a planar front obstruction, based on the Tietze–Nakajima theorem, handles the remaining 40. A third argument, a split-cover pinch, classifies every one of the 262,144 codes in the family of Example 3.22 of the same paper. Finally, the absorption criterion decides 63 of the 1,800 six-neuron codes that the published wheels census leaves open. The reduction counts, the certificates behind the 58 codes and the Example 3.22 classification, the wheel searches and the 63 census flags were each reproduced by a second, independent implementation, with stated controls.

Keywords: neural code, convex code, local obstruction, wheel, sprocket, simplicial complex, mandatory face, Tietze–Nakajima theorem.

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1 Introduction

A place cell in the hippocampus is active while the animal occupies a particular region of its surroundings, called the cell's *place field*. Write $U_1, \dots, U_n$ for the place fields of n cells. The cells active at a location x then form the set $c(x) = \{i : x \in U_i\}$, and the collection of all such sets is the *neural code* of the population. Since place fields are close to convex, a code arising from place cells ought to be *convex*, meaning that it can be realized by convex open sets. Deciding which codes are convex is a purely combinatorial problem about a finite family of subsets of $[n]$, and it has received sustained attention since Curto, Gross, Jeffries, Morrison, Omar, Rosen, Shiu and Youngs [5] posed it in this form. In dimension three and higher no algorithm is known, and Bukh and Jeffs [3] point out that the problem may be undecidable there. Only a handful of obstructions are known.

A *local obstruction* (Giusti and Itskov [6]; see [5, Theorem 1.3] for the form used here) is a face of the code's simplicial complex that is not a codeword and whose link is not contractible; convex codes have no local obstructions. A *wheel* [11] consists of three "spokes" and a "rim" arranged so that, in any realization by convex sets, some segment would have to cross a region it cannot enter; codes with a wheel are not convex. There are non-convex codes without local obstructions, for instance the sunflower codes of [7]. For these, [11] shows that the smallest one has a wheel and leaves the larger ones open. For codes with few maximal codewords, the expectation has been that these two obstructions are enough.

For codes with exactly four maximal codewords, Ahmed, Crepeau, Flores, Isekenegbe, Perez and Shiu [1] grouped such codes into 20 classes, L9–L28, according to the nerve of the maximal codewords. They proved that in the 15 classes L9–L23 a code is convex exactly when it has no local obstruction, and that a *minimal* code in L24 is convex when it has no wheel (a code is minimal when it coincides with the minimal code of its complex, and such a code has no local obstruction). Non-minimal L24 codes and the classes L25–L28 were left open, and the paper records the following conjecture of Jeffs as its Conjecture 1.1: *a code with up to four maximal codewords is convex if and only if it has no local obstructions and no wheels.* The same methods settle the minimal codes of L25, L26 and L27 (Section 9; the authors' group has obtained these cases independently). However, the L28 class still remains, in which all four maximal codewords share a common neuron. This report settles L28 for minimal codes and shows that the conjecture already fails for minimal codes.

Theorem 1.1 (main theorem)**.** *Up to renaming and duplication of neurons, there are exactly 58 minimal codes with four maximal codewords that have nerve L28, are not convex, and admit a realization with no wheel. Each of them is a counterexample to Conjecture 1.1 of [1]. Every other minimal L28 code is convex or has a sprocket.*

These 58 codes have from 7 to 14 neurons and either 12 or 14 codewords. Two new arguments prove non-convexity:

1. an *absorption lemma* for three convex open sets (Lemma 4.1), which through a criterion stated in terms of trunks, Lemma 4.2, handles 18 of the codes.

2. a *front obstruction* (Theorem 5.1), which handles the remaining 40 by cutting any convex realization down to a plane and applying the Tietze–Nakajima theorem to the "front" visible from a point.

The split-cover pinch (Theorem 7.3) also classifies the entire family extending Example 3.22 of [1] (Section 7). The absorption lemma yields a rule in the framework of Jeffs, Lienkaemper and Youngs [9], which resolves further cases in the published wheels census (Section 8).

Organization. After the preliminaries, Section 3 gives the finite reduction and classification (Figure 2), using published criteria and the results of Section 9. Sections 4–6 prove the main theorem. Sections 7 and 8 give further applications. Sections 10 and 11 cover verification and limitations.

2 Preliminaries

A code is $C \subseteq 2^{[n]}$ with $\emptyset \in C$ ([1], Assumption 2.5). Every neuron is assumed to lie in some codeword; a neuron that lies in no codeword can be deleted without changing convexity ([11, Proposition 6.3]), the simplicial complex or the minimal code. The simplicial complex $\Delta(C)$ of C is made up of all subsets of codewords, so the facets of $\Delta(C)$ are exactly the maximal codewords. A *realization* of C is a tuple $U = (U_1, \ldots, U_n)$ of open sets $U_i \subseteq \mathbb{R}^d$ with $C = \{c(x) : x \in \mathbb{R}^d\}$, where $c(x) = \{i : x \in U_i\}$ ([11, Section 2]); it is a *convex realization* if every U_i is convex, and C is *convex* (*open convex* in [4]) if it has a convex realization. The *atom* of a codeword c is $\{x : c(x) = c\}$. For $\varphi \subseteq [n]$ write $U_\varphi = \bigcap_{i\in\varphi} U_i$ (with $U_\emptyset = \mathbb{R}^d$) and $\mathrm{Tk}_C(\varphi) = \{c \in C : \varphi \subseteq c\}$, the *trunk*; the subscript is dropped when C is clear from context. Throughout, $\mathbb{B}(y, \rho)$ denotes the open ball of radius ρ about y.

Lemma 2.1 ([11, Lemma 4.1])**.** *Let U be a realization of C and let $\varphi, \psi_1, \ldots, \psi_r \subseteq [n]$. Then $U_\varphi \subseteq \bigcup_t U_{\psi_t}$ holds if and only if $\mathrm{Tk}_C(\varphi) \subseteq \bigcup_t \mathrm{Tk}_C(\psi_t)$.*

For example, if $\mathrm{Tk}(\varphi) = \mathrm{Tk}(\psi)$ then $U_\varphi = U_\psi$ in every realization, and if $\mathrm{Tk}(\varphi) = \emptyset$ then $U_\varphi = \emptyset$.

Local obstructions and minimal codes. A nonempty face σ of a complex Δ is *mandatory* if its link $\mathrm{Lk}_\Delta(\sigma)$ is not contractible. The *minimal code* of Δ is $C_{\min}(\Delta) = \{\text{facets}\} \cup \{\text{mandatory faces}\} \cup \{\emptyset\}$. A code C has no local obstruction iff $C_{\min}(\Delta(C)) \subseteq C$ ([1, Definition 2.22]); a convex code has no local obstruction ([5]; [1, Lemma 2.23]). Following [1, Theorem 3.19], C is *minimal* if $C = C_{\min}(\Delta(C))$. In particular, a minimal code has no local obstruction.

Max-intersection-complete codes. A code is *max-intersection-complete* when every nonempty intersection of two or more of its maximal codewords is itself a codeword. Two theorems of [4] are used repeatedly. Theorem 1.2 states that every max-intersection-complete code is convex, and Theorem 1.3 states that if C is open convex, then so is every D with $C \subsetneq D \subseteq \Delta(C)$.

Wheels. Following [11, Definition 3.1] ([1, Definition 2.14]), a tuple $W = (\sigma_1, \sigma_2, \sigma_3, \tau)$ of subsets of $[n]$ is a wheel of a realization U if (W(i)) $U_{\sigma_j} \cap U_{\sigma_k} = U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3} \neq \emptyset$ for all $j < k$; (W(ii)) $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3} \cap U_\tau = \emptyset$; and (W(iii)) if U_τ and each $U_{\sigma_j} \cap U_\tau$ are convex, then some segment with endpoints in $U_{\sigma_1} \cap U_\tau$ and $U_{\sigma_3} \cap U_\tau$ meets $U_{\sigma_2} \cap U_\tau$. W is a wheel *of the code* when it is a wheel of every realization. Codes with a wheel are not convex ([11, Theorem 3.4]). Whether W(i) and W(ii) hold depends only on trunks ([11, Proposition 4.3]), through the combinatorial conditions P(i) and P(ii) of [11, Definition 4.2]. Condition P(i) says that $\sigma_1 \cup \sigma_2 \cup \sigma_3 \in \Delta(C)$ and $\mathrm{Tk}(\sigma_j \cup \sigma_k) = \mathrm{Tk}(\sigma_1 \cup \sigma_2 \cup \sigma_3)$ for all $j < k$, and P(ii) says that $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau \notin \Delta(C)$. A tuple of faces of $\Delta(C)$ satisfying P(i), P(ii) and P(iii)°, the last meaning $\sigma_j \cup \tau \in \Delta(C)$ for $j = 1, 2, 3$, is a *partial wheel* ([11, Definition 4.2]; [1, Definition 2.16]). In [11, Section 4.2] (Definitions 4.4, 4.8 and 4.14), sprockets and wire wheels are the wheels that can be certified combinatorially, and wheel frames are the certified generators of wheels. When it introduces wheel frames, [11] also adopts that "it suffices to show that every realization has a wheel, which may vary from one realization to

another" ([11, Section 4.2, before Definition 4.14]). The paper [1] states only the first reading (its Definition 2.14), but it relies on the second when it attributes the non-convexity of its Example 2.32 to a wheel frame ([11, Proposition 4.16]) and when it cites [11, Theorem 3.8], a statement about wheel frames, as settling Conjecture 1.1 for codes on up to five neurons. Section 6 deals with both readings.

The nerve classes. For a 4-maximal code with facets $F_1, \dots, F_4$, the *nerve* is the complex on $\{1,2,3,4\}$ whose faces are the sets of indices whose facets intersect. Following [5, Table 2], [1] names the 20 possible nerves on exactly four vertices L9–L28 (the names L1–L8 belong to complexes on at most three vertices). L28 is the solid tetrahedron: every intersection of two or more facets, 11 in all, is nonempty (Figure 1).

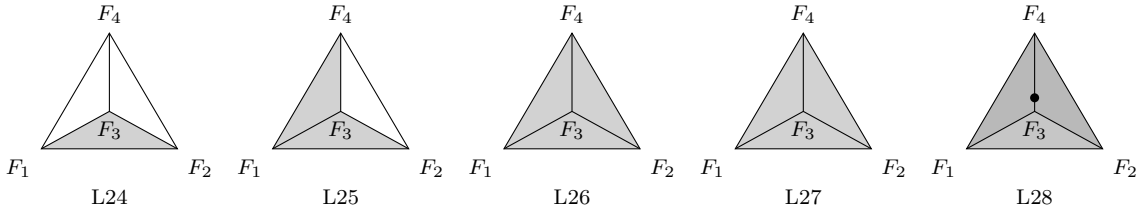


Figure 1: The five nerve classes relevant here, drawn on the vertices $F_1, \dots, F_4$. A shaded triangle indicates that those three facets share a neuron, and the solid tetrahedron of L28 indicates that all four do. L24: the 2-simplex $\{F_1, F_2, F_3\}$ together with three edges to F_4. L25: the minimal non-faces are $\{1,2,4\}, \{2,3,4\}$. L26: exactly one triple intersection is empty. L27: every triple intersection is nonempty but the quadruple one is empty. L28: all 11 intersections are nonempty. The classes follow [1, Figure 1] and [5, Table 2]; [1, Examples 2.37–2.38] identify L24 and L26 on examples.

Standard convexity facts. Let $X \subseteq \mathbb{R}^d$ be open and convex. The following are used below: (i) $x \in X$, $y \in \operatorname{cl} X$ imply $[x, y) \subseteq X$; (ii) if $g \notin X$ there is $n \neq 0$ with $\langle n, x - g\rangle < 0$ for all $x \in X$; (iii) for nonempty convex A, B, $\operatorname{conv}(A \cup B) = \bigcup_{a \in A, b \in B} [a, b]$; for $S \subseteq \mathbb{R}^d$, $\operatorname{cone}(S) = \{\lambda s : \lambda > 0,\ s \in S\}$, a convex cone when S is convex; (iv) (Tietze–Nakajima [12, 10]; modern statement in [2, Theorem 1]) a closed, connected, locally convex subset of $\mathbb{R}^d$ is convex.

3 Reduction and enumeration

3.1 Type reduction

Fix a minimal 4-maximal code C with facets $F_1, \dots, F_4$. The *type* of a neuron i is the set $T(i) = \{j : i \in F_j\}$ of indices of the facets containing it.

Lemma 3.1. *Every nonempty codeword of $C_{\min}(\Delta(C))$ is $F_S := \{i : S \subseteq T(i)\} = \bigcap_{j \in S} F_j$ for some nonempty $S \subseteq \{1,2,3,4\}$. In particular, two neurons i, i' with $T(i) = T(i')$ belong to the same codewords.*

Proof. The nonempty codewords of $C_{\min}(\Delta(C))$ are the facets $F_j = F_{\{j\}}$ and the mandatory faces of $\Delta(C)$. Every mandatory face is either a facet or an intersection of facets by [1, Lemma 2.24], and the definition of type gives $\bigcap_{j \in S} F_j = F_S$. The second statement holds because $i \in F_S$ iff $S \subseteq T(i)$, and because the empty codeword contains neither i nor i'. □

Lemma 3.2. *If two neurons have the same type, deleting one of them neither creates nor destroys convexity.*

Proof. Dropping U_i from a convex realization of C gives a convex realization of the smaller code. In the other direction, take a convex realization of the smaller code and put $U_i := U_{i'}$; Lemma 3.1 shows that the resulting code is exactly C. □

Theorem 3.3 (type reduction)**.** *A minimal 4-maximal code is convex if and only if its type reduction is convex. The type reduction keeps one neuron for each occupied type, so it has at most 15 neurons, and at most 14 unless the nerve is L28.*

Proof. Apply Lemma 3.2 to one pair of neurons of equal type at a time. Each deletion preserves convexity in both directions, and the deleted code inherits the property that neurons of equal type lie in the same codewords, so the deletions can be repeated until every occupied type is represented by exactly one neuron. There are $2^4 - 1 = 15$ nonempty types, and the type $\{1, 2, 3, 4\}$ is occupied only when the nerve is L28. □

As a result, a question about codes on arbitrarily many neurons becomes a finite enumeration (Figure 2).

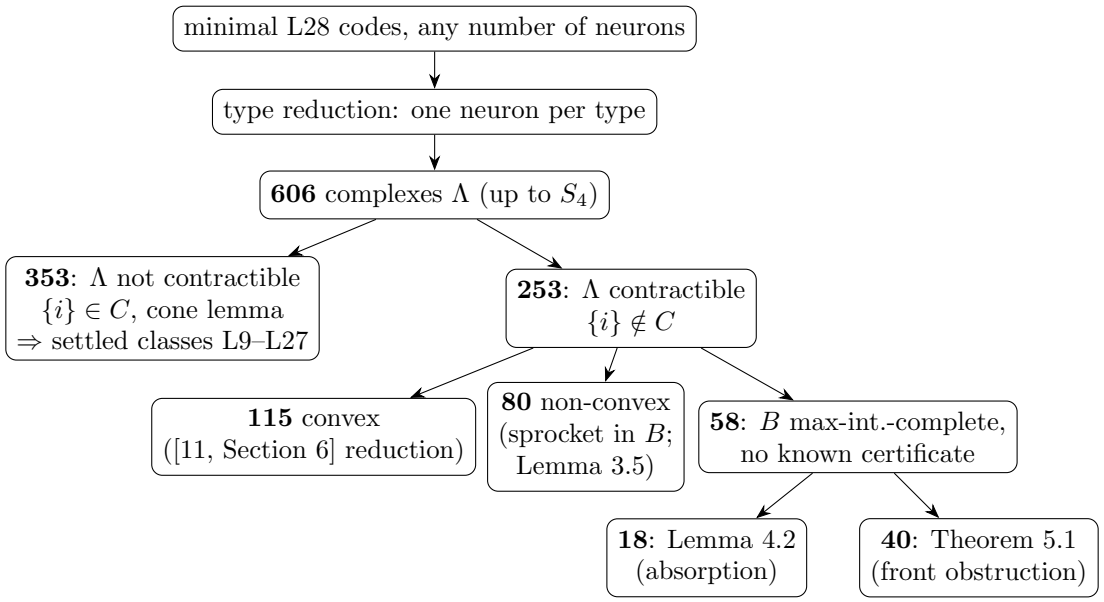


Figure 2: Outline of the reduction. Here i is the apex neuron, $\Lambda = \mathrm{Lk}(\{i\})$ is its link in $\Delta(C)$, and $B = C_{\min}(\Lambda)$ is the deconed code of Section 3.4. An independent implementation recomputed every count (Section 10). Contractibility was certified in both directions, and no case remained undetermined.

3.2 Cone lemma

Let C be a minimal L28 code and $K = F_1 \cap F_2 \cap F_3 \cap F_4 \neq \emptyset$. Every neuron of K has type $\{1, 2, 3, 4\}$, so the type reduction replaces K by one neuron i, called the *apex*. Then $\Delta(C) = i * \Lambda$, where Λ is a complex on the remaining neurons.

Lemma 3.4 (cone lemma)**.** *Suppose a neuron i of a code C belongs to every nonempty codeword and $\{i\} \in C$, and let $B = C|_{[n]\setminus\{i\}}$. Then C is convex iff B is convex.*

Proof. ($\Leftarrow$) Take a realization of B by convex sets V_j contained in an open ball Ω, and set $U_i = \Omega$ and $U_j = V_j$. Inside Ω the codewords are $\{i\} \cup c$ with $c \in B$, where $c = \emptyset$ gives $\{i\}$; outside Ω the

codeword is $\emptyset$. ($\Rightarrow$) This is [11, Lemma 6.1]: restricting a convex code to a subset of its neurons gives a convex code. □

Lemma 3.5 (sprockets lift to cones)**.** *Let B be a code on $[n]\setminus\{i\}$, and let C be either $\{\emptyset\}\cup\{\rho\cup\{i\} : \emptyset \neq \rho \in B\}$ or that code with $\{i\}$ added. If $(\sigma_1, \sigma_2, \sigma_3, \tau)$ is a sprocket of B with witnesses ρ_1, ρ_3, then it is a sprocket of C with the same witnesses.*

Proof. The map $\rho \mapsto \rho \cup \{i\}$ is one-to-one on $B \setminus \{\emptyset\}$, and for every nonempty $\varphi \subseteq [n] \setminus \{i\}$ it carries $\mathrm{Tk}_B(\varphi)$ onto $\mathrm{Tk}_C(\varphi)$, because neither $\emptyset$ nor $\{i\}$ contains φ. Hence, for nonempty subsets of $[n] \setminus \{i\}$, an equality or inclusion among trunks and unions of trunks holds in B iff it holds in C, and $\varphi \in \Delta(C)$ iff $\mathrm{Tk}_C(\varphi) \neq \emptyset$ iff $\varphi \in \Delta(B)$. The conditions P(i), P(ii), P(iii)° and S(1)–S(3) of [1, Definition 2.16] are all of this form once the sets whose trunks are compared are nonempty. The only exceptions are sets on the right-hand side of an inclusion, where $\mathrm{Tk}(\emptyset)$ is the whole code and the inclusion is trivial in both. Now $\tau \neq \emptyset$, since P(i) puts $\sigma_1 \cup \sigma_2 \cup \sigma_3$ in $\Delta(B)$ while P(ii) keeps $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \tau$ out of it; and at most one spoke is empty, since $\sigma_j = \sigma_k = \emptyset$ would give $\mathrm{Tk}_B(\emptyset) = B = \mathrm{Tk}_B(\sigma_l)$, forcing $\sigma_l = \emptyset$ because $\emptyset \in B$ lies in no other trunk, and then P(ii) would contradict P(iii)°. So $\sigma_j \cup \sigma_k$, $\sigma_j \cup \tau$, τ and $\rho_1 \cup \rho_3 \cup \tau$ are nonempty, and the only sets that may be empty, ρ_1, ρ_3 in S(1)–S(2) and σ_2 in S(3), stand on right-hand sides. (Compare [1, Remark 3.23].) The same argument, with $\rho \mapsto \rho \cup \{i\}$ replaced by the bijection $c \mapsto c \cap N$ from a minimal 4-maximal code onto its type reduction on a set N of one neuron per type (Lemma 3.1), shows that a sprocket of the type reduction is a sprocket of the code. □

By [1, Definition 2.22], $\{i\} \in C_{\min}(\Delta(C))$ iff $\mathrm{Lk}(\{i\}) = \Lambda$ is not contractible. When Λ is not contractible, the cone lemma applies, so C is convex iff $C_{\min}(\Lambda)$ is. Removing the apex empties the quadruple intersection, so $C_{\min}(\Lambda)$ is a minimal 4-maximal code whose nerve is one of L9–L27: its four facets are nonempty, distinct and pairwise non-nested, and their quadruple intersection is empty. All of these classes are settled for minimal codes, by [1, Theorem 3.1] for L9–L23, by [1, Theorem 3.19] for L24, and by the same methods for L25–L27 (Section 9). Moreover, if $C_{\min}(\Lambda)$ is not convex, then it has a sprocket (by [1, Theorem 3.19] for L24 and Section 9 for L25 and L26, since minimal L9–L23 and L27 codes are convex), and by Lemma 3.5 that sprocket is also a sprocket of C. When Λ is contractible, $\{i\} \notin C$ and the cone lemma is not available.

3.3 Enumeration

The complex Λ is determined by which of the 14 nonempty proper subsets of $\{1, 2, 3, 4\}$ occur as types, subject to the four facets obtained by removing the apex being nonempty, distinct and pairwise non-nested. Up to the action of the symmetric group S_4, **606** complexes remain. For each of them contractibility was decided with a certificate in both directions: nonvanishing reduced homology shows that Λ is not contractible, and an explicit sequence of elementary collapses shows that it is. Acyclicity was never used as a substitute. The outcome is **353** non-contractible complexes, **253** contractible ones and **0** undetermined. Among the 253, the nerve of Λ is either L26 (168 cases) or lacks at least one edge, which means that two of the deconed facets are disjoint (85 cases); L24, L25 and L27 never occur.

3.4 Contractible cases

Throughout, $B = C_{\min}(\Lambda)$, and hence $C = \{\emptyset\} \cup \{\rho \cup \{i\} : \emptyset \neq \rho \in B\}$.

Eighty are non-convex. In 80 of the 253 cases, B has an explicit sprocket. By Lemma 3.5 the same tuple, with the same witnesses, is a sprocket of C, so C is non-convex by [1, Lemma 2.18].

(Alternatively, B is non-convex, so Lemma 3.4 shows that the cone $D = C \cup \{\{i\}\}$ is non-convex, and since $C \subsetneq D \subseteq \Delta(C)$, [4, Theorem 1.3] gives the same conclusion.) These 80 are the cases where B lies in L26 and has one containment $F_a \cap F_b \subseteq F_c$, in line with the dichotomy of Section 9; the argument itself, however, relies only on the sprockets that were found.

One hundred fifteen are convex. Their convexity is certified with results of [11]. Deleting a redundant neuron preserves convexity in both directions ([11, Proposition 6.3]), and after such deletions each of the 115 codes is decomposable in the sense of [11, Definition 6.4]. Proposition 6.8 of [11] is used, with top-dimensional convexity as in [11, Definition 6.7], in the form that its proof establishes: *if a decomposable code C' has a top-dimensionally convex ambient code $C'|_{[m]\setminus\varphi}$ and a convex embedded code $C'|_{\varphi}$, then C' is convex.* The printed statement places the top-dimensional hypothesis on the embedded code, but the proof uses a top-dimensional realization of the ambient code to place a ball inside the atom of the ambient codeword ψ. For 89 of the 115 codes there is a decomposition whose embedded code is max-intersection-complete and whose ambient code is either max-intersection-complete (possible for 86 of them) or, for the other 3, a code on five neurons with no local obstruction and no wheel frame. Such a code is top-dimensionally convex by [11, Lemma 6.9] or by [11, Theorem 3.8 and Proposition 6.13]. For each of the remaining 26 codes, one neuron that lies in a single codeword is split off. The ambient code then contains the minimal code of its simplicial complex, and after redundant neurons are deleted this minimal code becomes a code on five neurons with no local obstruction and no wheel frame. That code is top-dimensionally convex by [11, Theorem 3.8 and Proposition 6.13], and hence so is the ambient code, by [11, Lemma 6.10(i)] and the following fact, which is not stated in [11]: if j is a redundant neuron of a code C', with $\mathrm{Tk}_{C'}(\{j\}) = \mathrm{Tk}_{C'}(\sigma)$ and $j \notin \sigma$, and the code obtained from C' by deleting j is top-dimensionally convex, then so is C', because setting $U_j := U_\sigma$ as in the proof of [11, Proposition 6.3] changes no atom. The procedure was implemented independently and checked against the published data of [11]. On all 210 four-facet records of that census it reproduces the census's own `reduced` and `indecomposable` flags, and it never declares convex a code that the census marks as having a wheel. A consistency check, "B non-convex implies C not certified convex," passes on all 253.

In each of the remaining 58 cases, B is max-intersection-complete and hence convex by [4, Theorem 1.2]. So $D = C \cup \{\{i\}\}$ is convex, and C differs from this convex code only by the missing singleton $\{i\}$. The 58 come in two shapes, with 18 and 40 codes respectively (Table 1).

Table 1: The 58 codes, grouped by shape, by the number of neurons remaining after type reduction, and by the number of codewords.

neurons	7	8	9	10	11	12	13	14	total	codewords
first shape (Lemma 4.2)	1	3	5	5	3	1	–	–	18	12
second shape (Theorem 5.1)	1	3	6	10	10	6	3	1	40	14

4 Absorption obstruction

4.1 Absorption lemma

Lemma 4.1 (absorption lemma)**.** *Suppose $A, B, M \subseteq \mathbb{R}^d$ are open convex sets such that $A \cap M = B \cap M =: G \neq \emptyset$. Then* $\mathrm{conv}(A \cup B) \cap M = G$.

Proof. For $\subseteq$, use (iii) to write a point of the left side as $z \in [a, b] \cap M$ with $a \in A$ and $b \in B$, and assume $z \notin G$. As $z \in M$, this means $z \notin A$ and $z \notin B$, so $z \neq a, b$. Choose $s \in G$. Then

$[s,z] \subseteq M$, and therefore

$$[s,z] \cap A = [s,z] \cap M \cap A = [s,z] \cap G = [s,z] \cap M \cap B = [s,z] \cap B.$$

Because A is open and convex, $s \in A$ and $z \notin A$, this common set has the form $[s,g^*)$ for a single point $g^* = s + r(z-s)$ with $r \in (0,1]$, and $g^* \in \partial A \cap \partial B \cap M$.

Case $r = 1$ ($g^* = z$). Here $z \in \operatorname{cl} A \cap \operatorname{cl} B$, and (i) gives $[a,z) \subseteq A$ and $[b,z) \subseteq B$. Since M is open and contains z, points $a' \in [a,z)$ and $b' \in [b,z)$ sufficiently near z lie in M, and hence in G. As z lies strictly between a and b, convexity of G gives $z \in [a',b'] \subseteq G$, a contradiction.

Case $r < 1$. Choose $\rho > 0$ with $\mathbb{B}(g^*,\rho) \subseteq M$. Then $A \cap \mathbb{B}(g^*,\rho) = G \cap \mathbb{B}(g^*,\rho) = B \cap \mathbb{B}(g^*,\rho)$, so A and B agree near g^*. If X is open and convex with $g^* \in \operatorname{cl} X$, then (i) gives $(g^*,x] \subseteq X$ for all $x \in X$, and consequently $\operatorname{cone}(X - g^*) = \operatorname{cone}((X \cap \mathbb{B}(g^*,\rho)) - g^*)$. It follows that $K := \operatorname{cone}(A - g^*) = \operatorname{cone}(B - g^*)$ is a convex cone with $A \cup B \subseteq g^* + K$. Hence $\operatorname{conv}(A \cup B) \subseteq g^* + K$, and in particular $z - g^* \in K$. Moreover $w := s - g^* \in \operatorname{int} K$, because A is open, $s \in A$ and $s \neq g^*$; and $z - g^* = -\frac{1-r}{r}\, w$ with $\frac{1-r}{r} > 0$. A convex cone that contains an interior point w together with a negative multiple of w must be all of $\mathbb{R}^d$. Indeed $-w \in K$, and for any x we have $w + x/\lambda \in \operatorname{int} K$ once λ is large, so $x = (\lambda w + x) + (-\lambda w) \in K$. On the other hand $g^* \notin A$, so (ii) gives $K \subseteq \{\langle n, \cdot \rangle < 0\} \neq \mathbb{R}^d$, another contradiction. □

Figure 3 illustrates the argument.

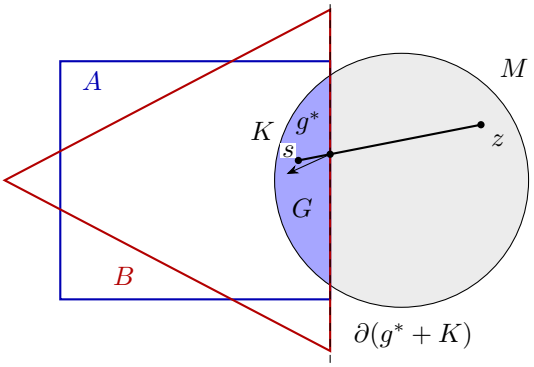


Figure 3: The absorption lemma. The blue rectangle A and the red triangle B meet the disk M in the same shaded set G, the part of M to the left of the dashed line. If a segment from A to B contained a point $z \in M \setminus G$, then along $[s,z]$ with $s \in G$ there would be a boundary point g^* near which A and B agree, so that $\operatorname{cone}(A - g^*) = \operatorname{cone}(B - g^*) = K$. In the picture, $g^* + K$ is the open half-plane to the left of the dashed line. The inclusion $\operatorname{conv}(A \cup B) \subseteq g^* + K$ would then give $z - g^* \in K$. But $z - g^*$ is a negative multiple of the interior direction $s - g^*$, which would force $K = \mathbb{R}^d$, contradicting the separating hyperplane at g^*.

4.2 Trunk criterion

Lemma 4.2 (Lemma Z). *Let C be a code, and let $\sigma, k_3, k_4, a, a', b, b'$ be faces of $\Delta(C)$ such that*

(Z1) $\mathrm{Tk}(a), \mathrm{Tk}(a') \subseteq \mathrm{Tk}(k_4)$ *and* $\mathrm{Tk}(b), \mathrm{Tk}(b') \subseteq \mathrm{Tk}(k_3)$*;*

(Z2) $\mathrm{Tk}(a \cup k_3) = \mathrm{Tk}(a \cup b) = \mathrm{Tk}(b \cup k_4)$ *and* $\mathrm{Tk}(a' \cup k_3) = \mathrm{Tk}(a' \cup b') = \mathrm{Tk}(b' \cup k_4)$*;*

(Z3) $\mathrm{Tk}(a \cup b') = \mathrm{Tk}(a' \cup b) = \emptyset$*;*

(Z4) $\mathrm{Tk}(a \cup a')$, $\mathrm{Tk}(b \cup b')$, $\mathrm{Tk}(a \cup b)$, $\mathrm{Tk}(a' \cup b')$ *are all nonempty;*

(Z5) $\mathrm{Tk}(a \cup a') \cup \mathrm{Tk}(b \cup b') \subseteq \mathrm{Tk}(\sigma) \subseteq \mathrm{Tk}(k_3) \cup \mathrm{Tk}(k_4)$.

Then C is not convex.

Proof. Assume that U is a convex realization, and set $K_3 = U_{k_3}$, $K_4 = U_{k_4}$, $A = U_a$, $A' = U_{a'}$, $B = U_b$, $B' = U_{b'}$, $M = K_3 \cap K_4$, $G = A \cap B$, $G' = A' \cap B'$. By Lemma 2.1: (Z1) gives $A, A' \subseteq K_4$ and $B, B' \subseteq K_3$; (Z2) gives $A \cap K_3 = G = B \cap K_4$, hence $A \cap M = G = B \cap M$, and likewise $A' \cap M = G' = B' \cap M$; (Z3) gives $A \cap B' = A' \cap B = \emptyset$, so $G \cap G' = \emptyset$; (Z4) gives $p \in A \cap A'$, $q \in B \cap B'$; (Z5) gives $[p, q] \subseteq U_\sigma \subseteq K_3 \cup K_4$. Note that $p \notin K_3$, since otherwise $p \in A \cap K_3 = G \subseteq B$ and so $p \in A' \cap B = \emptyset$; by symmetry $q \notin K_4$. The sets $[p, q] \cap K_3 \ni q$ and $[p, q] \cap K_4 \ni p$ are relatively open and cover the connected segment, so they have a common point $z \in M$. Applying Lemma 4.1 to (A, B, M) gives $z \in G$, and applying it to (A', B', M) gives $z \in G'$. This is impossible because $G \cap G' = \emptyset$. □

4.3 Certified codes

Each code of the first shape has faces satisfying (Z1)–(Z5). The smallest of them has seven neurons and 12 codewords. Naming the neurons by their types, with apex i, it is

$$\emptyset; \quad \{123, i\},\ \{234, i\}; \quad \{12, 123, i\},\ \{13, 123, i\},\ \{24, 234, i\},\ \{34, 234, i\};$$
$$\{123, 234, i\},\ \{12, 13, 123, i\},\ \{24, 34, 234, i\}; \quad \{12, 24, 123, 234, i\},\ \{13, 34, 123, 234, i\},$$

and its maximal codewords are the last two listed together with $\{12, 13, 123, i\}$ and $\{24, 34, 234, i\}$. A search for certificates in which every slot is a single neuron finds them for all 18 codes: four ordered tuples $(\sigma, k_3, k_4, a, a', b, b')$ per code, 72 in all, which form a single orbit per code under the two relabelings $(a, b) \leftrightarrow (a', b')$ and $(k_3, a, a') \leftrightarrow (k_4, b, b')$ that preserve (Z1)–(Z5). The same search finds none for the 58 coned codes $C \cup \{\{i\}\}$, which are convex. The configuration is sketched in Figure 4.

5 Front obstruction

5.1 Core code

Consider the code on the seven neurons i, a, b, c, x, y, z with the following 14 codewords:

$$\emptyset; \quad ia,\ ib,\ ic; \quad iab,\ iac,\ ibc,\ iax,\ iby,\ icz; \quad iabc,\ iabxy,\ iacxz,\ ibcyz.$$

This code is denoted C_0. All of its maximal codewords $iabc, iabxy, iacxz, ibcyz$ contain i, so its nerve is L28, and $C_0 = C_{\min}(\Delta(C_0))$, so it is minimal. The whole argument rests on the following

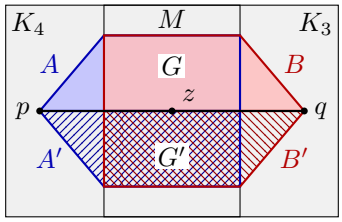


Figure 4: The configuration of Lemma 4.2. Here $A, A' \subseteq K_4$ and $B, B' \subseteq K_3$. The sets A (blue) and B (red) meet $M = K_3 \cap K_4$ in a common set G above the segment $[p, q]$, and A' and B' (hatched) meet it in a common set G' below the segment. A segment inside $U_\sigma \subseteq K_3 \cup K_4$ from $p \in A \cap A'$ to $q \in B \cap B'$ has to cross M at some point z, and the absorption lemma places z in both G and G', which the hypotheses make disjoint. Because of this contradiction no drawing by open convex sets can be validated: here the sets are closed polygons, and G and G', like A and B' and like A' and B, meet along the segment through p, z and q. In the seven-neuron code above, $\sigma = \{i\}$, $k_3 = \{123\}$, $k_4 = \{234\}$, $a = \{34\}$, $a' = \{24\}$, $b = \{13\}$, $b' = \{12\}$.

trunk identities:

$$\mathrm{Tk}(x) \subseteq \mathrm{Tk}(a), \qquad \mathrm{Tk}(y) \subseteq \mathrm{Tk}(b), \qquad \mathrm{Tk}(z) \subseteq \mathrm{Tk}(c);$$
$$\mathrm{Tk}(ay) = \mathrm{Tk}(xy) = \mathrm{Tk}(bx) = \{iabxy\}, \qquad \mathrm{Tk}(az) = \mathrm{Tk}(xz) = \mathrm{Tk}(cx) = \{iacxz\},$$
$$\mathrm{Tk}(bz) = \mathrm{Tk}(yz) = \mathrm{Tk}(cy) = \{ibcyz\};$$
$$\mathrm{Tk}(xyz) = \emptyset, \qquad \mathrm{Tk}(abc) = \{iabc\}, \qquad \mathrm{Tk}(abcs) = \emptyset \quad (s \in \{x, y, z\}).$$

Given a realization, set $R_1 = U_a \cap U_b \cap U_c$, $R_2 = U_x \cap U_y$, $R_3 = U_x \cap U_z$ and $R_4 = U_y \cap U_z$. By Lemma 2.1, $U_x \subseteq U_a$ and $U_a \cap U_y = R_2 = U_b \cap U_x$, together with the cyclic analogues. Moreover $U_x \cap U_y \cap U_z = \emptyset$, and R_1 is nonempty and disjoint from $U_x \cup U_y \cup U_z$. Informally, the three "sticks" U_x, U_y, U_z meet two at a time at three corners but have no point in common, while the three "lobes" $U_a \supseteq U_x$ and so on share a point in R_1, yet each lobe meets the other two sticks only inside its own stick (Figure 5).

5.2 Non-convexity proof

Theorem 5.1 (front obstruction)**.** *Let C be a code, and let $x, y, z, a, b, c \in \Delta(C)$ be faces satisfying*

(F1) $\mathrm{Tk}(x) \subseteq \mathrm{Tk}(a)$, $\mathrm{Tk}(y) \subseteq \mathrm{Tk}(b)$, $\mathrm{Tk}(z) \subseteq \mathrm{Tk}(c)$*;*

(F2) $\mathrm{Tk}(a \cup y) = \mathrm{Tk}(x \cup y) = \mathrm{Tk}(b \cup x)$, $\mathrm{Tk}(a \cup z) = \mathrm{Tk}(x \cup z) = \mathrm{Tk}(c \cup x)$, $\mathrm{Tk}(b \cup z) = \mathrm{Tk}(y \cup z) = \mathrm{Tk}(c \cup y)$*;*

(F3) $\mathrm{Tk}(x \cup y \cup z) = \emptyset$ *and* $\mathrm{Tk}(x \cup y), \mathrm{Tk}(x \cup z), \mathrm{Tk}(y \cup z) \neq \emptyset$*;*

(F4) $\mathrm{Tk}(a \cup b \cup c) \neq \emptyset$ *and* $\mathrm{Tk}(a \cup b \cup c \cup s) = \emptyset$ *for* $s \in \{x, y, z\}$*;*

(F5) $\mathrm{Tk}(x \cup y) = \{F_{xy}\}$, $\mathrm{Tk}(x \cup z) = \{F_{xz}\}$, $\mathrm{Tk}(y \cup z) = \{F_{yz}\}$*; every codeword of* $C \setminus \mathrm{Tk}(a \cup b \cup c)$ *lies in one of* F_{xy}, F_{xz}, F_{yz}*; and* $\Delta_A := \langle F_{xy}, F_{xz}, F_{yz} \rangle$ *has a mandatory face* $\sigma \notin C$*.*

Then C is not convex.

The code C_0 satisfies these hypotheses with the six named neurons in these roles and $\sigma = \{i\}$.

Proof. Step A (reduction to the plane). Let $\{U_j\}$ be a convex realization in $\mathbb{R}^d$. First, $d \geq 2$: when $d = 1$, three open intervals that meet pairwise have a common point, which would contradict

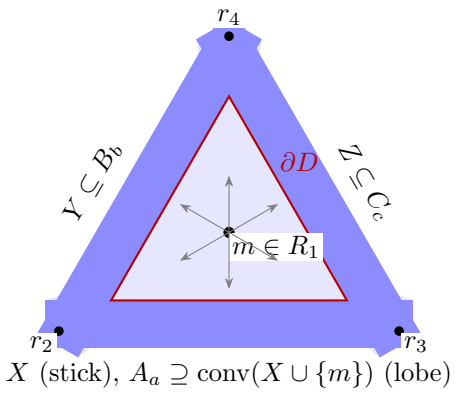


Figure 5: The core code C_0 drawn in the plane; Step B shows that this configuration cannot exist. The three dark "sticks" X, Y, Z run along the sides of the corner triangle $r_2r_3r_4$; any two of them meet at a corner, but all three never meet. Each light lobe contains its own stick and the point $m \in R_1$, and meets the other two sticks only inside its own stick. Every ray from m hits a stick, and a ray that hits two sticks enters both at the same depth (B2). The entry points make up the front ∂D (red), which consists of the inner edges of the sticks. The region E outside the front is closed, connected and locally convex, so by Tietze–Nakajima it is convex and hence equal to the whole plane, but E does not contain m. Near each corner a drawn lobe overlaps a neighbouring stick outside its own stick, and a ray from m that passes beside a corner enters one stick before the other, which B1 and B2 forbid.

$U_x \cap U_y \cap U_z = \emptyset$. Intersecting all the sets with an open ball that meets every atom does not change the code and makes every set bounded. Pick affinely independent points $r_2 \in R_2$, $r_3 \in R_3$, $r_4 \in R_4$, and let A be the plane through them. The sets $U_j \cap A$ form a convex realization, in $A \cong \mathbb{R}^2$, of the code $C' = \{c \in C : \mathrm{atom}(c) \cap A \neq \emptyset\} \subseteq C$, and C' contains F_{xy}, F_{xz}, F_{yz} and $\emptyset$.

Suppose $A \cap R_1 = \emptyset$. Then no codeword of C' contains $a \cup b \cup c$, so (F5) puts every codeword of C' inside a corner codeword, and therefore $\Delta(C') = \Delta_A$. Since a convex code contains every mandatory face of its complex, $\sigma \in C' \subseteq C$, which contradicts (F5). (For C_0, the link of $\{i\}$ in $\langle iabxy, iacxz, ibcyz\rangle$ consists of three tetrahedra that meet pairwise in the edges ax, by, cz and have empty triple intersection. It is homotopy equivalent to a circle, so $\{i\}$ is mandatory, while $\{i\} \notin C_0$.) The six sets $X = U_x \cap A$, Y, Z, $A_a = U_a \cap A$, B_b, C_c are then open convex subsets of the plane that satisfy all the identities, with $R_1 \cap A \neq \emptyset$ and $r_j \in R_j \cap A$.

Step B (no planar configuration). Fix $m \in R_1 \cap A$. Because R_1 is open and disjoint from the sticks, $m \notin \mathrm{cl}\, X, \mathrm{cl}\, Y, \mathrm{cl}\, Z$.

B1. Here $\mathrm{conv}(X \cup \{m\}) \subseteq A_a$ and $A_a \cap Y = X \cap Y$, hence $\mathrm{conv}(X \cup \{m\}) \cap (Y \setminus X) = \emptyset$, and the same holds for the five symmetric pairs.

B2 (same entry). Suppose a ray from m meets X in the open interval of depths (s_X, f_X) (where $f_X = \infty$ is possible) and meets Y in (s_Y, f_Y). A point of the ray at depth $t < f_X$ lies on a segment from m to a point of X, so it lies in $\mathrm{conv}(X \cup \{m\})$. By B1, if it lies in Y then it lies in X, that is, $t > s_X$. Thus $s_Y \geq s_X$ unless $s_Y \geq f_X$, and symmetrically $s_X \geq s_Y$ unless $s_X \geq f_Y$. If $s_Y \geq f_X$, then $s_Y > s_X$, so the second statement gives $s_X \geq f_Y$, and then $s_Y \geq f_X > s_X \geq f_Y > s_Y$; symmetrically, $s_X \geq f_Y$ is impossible. Hence $s_Y < f_X$ and $s_X < f_Y$, and the two statements give $s_X = s_Y$.

B3. No ray meets all three sticks, since it would enter them at a single depth and would then run through $X \cap Y \cap Z = \emptyset$.

B4 (m is inside the corner triangle). Choose r_2, r_3, r_4 again, inside A and in general position together with m. This can be done without changing the plane, because each $R_j \cap A$ is a nonempty open subset of A and the degenerate choices form a proper algebraic subset of the open set $(R_2 \cap$

$A) \times (R_3 \cap A) \times (R_4 \cap A)$. The segments then satisfy $mr_2 \subseteq A_a \cap B_b$, $mr_3 \subseteq A_a \cap C_c$, $mr_4 \subseteq B_b \cap C_c$, $r_2r_3 \subseteq X$, $r_2r_4 \subseteq Y$ and $r_3r_4 \subseteq Z$. Four points in general position are either in convex position or have one of them inside the triangle formed by the others. If $r_2 \in \triangle(m, r_3, r_4) \subseteq C_c$, then $r_2 \in C_c \cap Y = R_4 \subseteq Z$ is a point of $X \cap Y \cap Z$. In convex position the diagonals cross at a point of $(A_a \cap C_c) \cap Y = R_2 \cap R_4$, of $(A_a \cap B_b) \cap Z = R_3 \cap R_4$ or of $(B_b \cap C_c) \cap X = R_2 \cap R_3$, and each of these lies in $X \cap Y \cap Z$. Hence m lies in the interior of $\triangle r_2r_3r_4$.

B5–B6 (the front). Each ray from m leaves the triangle through an edge and therefore meets a stick. Let S_X be the open set of directions whose rays meet X, and let s_X be the entry depth. Then s_X is continuous on S_X: it is upper semicontinuous because X is open. It is lower semicontinuous because, by (i), $\operatorname{cl} X$ meets the ray in direction d only at depths $t \geq s_X(d)$. So for $0 < \varepsilon < s_X(d)$ the compact segment $\{m + td : 0 \leq t \leq s_X(d) - \varepsilon\}$ has positive distance from $\operatorname{cl} X$, the same segment in every nearby direction misses $\operatorname{cl} X$, and $s_X > s_X(d) - \varepsilon$ there. For a direction d, let $\Phi(d)$ be the smallest entry depth among the sticks that the ray in direction d meets. By B2, $\Phi = s_X$ everywhere on S_X, and similarly on S_Y, S_Z. These open sets cover the circle, so Φ is continuous and positive. The set $D = \{m + td : 0 \leq t < \Phi(d)\}$ is therefore a bounded open set containing m, and $E = \mathbb{R}^2 \setminus D$ is closed. It is also connected, being the image of $S^1 \times [0, \infty)$ under the continuous map $(d, t) \mapsto m + (\Phi(d) + t)d$.

B7 (E is locally convex). Let $p = m + \Phi(d)d$ with $d \in S_X$, and take an open ball $N \ni p$ that avoids m and is small enough that the direction of every point of N lies in S_X and every depth in N is less than the exit depth of X (the exit depth is lower semicontinuous on S_X). For $q \in N$ we have $q \in E$ iff $\operatorname{depth}(q) \geq s_X(\operatorname{dir} q)$ iff $q \in \operatorname{cl} X$, since on any line meeting the open convex set X, $\operatorname{cl} X$ meets the line in the closure of $X \cap \text{line}$. So $E \cap N = \operatorname{cl} X \cap N$, which is convex. At interior points of E there is nothing to prove.

B8. By (iv), every closed, connected, locally convex subset of $\mathbb{R}^2$ is convex. Since E contains the complement of a bounded set, a convex E would have to be all of $\mathbb{R}^2$, but $m \notin E$. □

5.3 Certified codes

Each of the 40 codes of the second shape contains, besides the apex, neurons of the six types $a = 123$, $b = 124$, $c = 134$, $x = 23$, $y = 24$, $z = 34$, and its restriction to those seven neurons is C_0. By [11, Lemma 6.1], each is therefore non-convex. Each is also minimal, so it has no local obstruction. When the criterion (F1)–(F5) is searched over the lattice of trunks of every code, it fires on all 40, with C_0 as the seven-neuron member, and on none of 2,997 codes known to be convex (Section 10). It does not fire on the 18 codes of Section 4, and the absorption criterion does not fire on the 40, so the two mechanisms are distinct. Let B_0 denote the code induced by C_0 on its six non-apex neurons; it has 14 codewords, and $C_0 \cup \{\{i\}\}$ is the cone $i * B_0$. The code B_0 is max-intersection-complete and has an explicit convex polyhedral realization in $\mathbb{R}^3$ (19 planes, with the code checked exactly). A convex realization of B_0 in the plane, on the other hand, would satisfy all the identities with $R_1 \neq \emptyset$ (because $abc \in B_0$), and Step B of Theorem 5.1 rules this out. Hence the open embedding dimension of B_0 is exactly 3, and by the cone lemma the same is true of $C_0 \cup \{\{i\}\}$ (Figure 6; embedding dimension is meant in the sense of [8]).

Minimum size. Every code that satisfies (F1)–(F5) and has no local obstruction has at least seven neurons, so C_0 is a smallest such code and none lies in the six-neuron census of [11]. (Without the condition on local obstructions there are instances on four neurons, all of which have local obstructions.) To see this, note that F_{xy} is a facet of $\Delta(C)$, since it is the only codeword containing $x \cup y$; likewise F_{xz} and F_{yz} are facets. The three are distinct by (F3), and by (F5) every other facet contains $a \cup b \cup c$. By (F3) and (F4), the face $F_{xy} \cap F_{xz} \supseteq x \neq \emptyset$ lies in no other facet, so it



$w = 0$: the three sticks; no common point

$w = 2$: the three lobes; they overlap in $R_1 \ni m$

Figure 6: Two horizontal slices through the exact convex polyhedral realization in $\mathbb{R}^3$ of B_0, the code that C_0 induces on its six non-apex neurons (19 planes; the coordinates were computed exactly from the half-space description and are rounded here for display). The sticks are thin prisms lying along the sides of a triangle inside the slab $|w| < \frac{1}{20}$. Each lobe has the same four side faces as its stick, with the inner one tilted, but it runs from the bottom of the slab (for two lobes, at most $\frac{1}{500}$ below it) up to a height of about 3, so that the three lobes meet at height $w = 2$. The identities of Section 5 hold because a lobe can reach a neighbouring stick only through the four faces of its own prism. Step B of Theorem 5.1 shows that no configuration of this kind exists in a plane, so the open embedding dimension of B_0 is exactly 3.

is mandatory by [1, Lemma 2.26] and hence lies in C. The same holds for the other two pairwise intersections and for the three facets. By [1, Lemma 2.24], the face $\sigma \notin C$ of (F5) is therefore $I := F_{xy} \cap F_{xz} \cap F_{yz}$. Some codeword $c^* \supseteq a \cup b \cup c$ contains I, since otherwise $\mathrm{Lk}_{\Delta(C)}(I) = \mathrm{Lk}_{\Delta_A}(I)$ would be non-contractible and I would be a local obstruction. The seven sets I, $a \setminus F_{yz}$, $b \setminus F_{xz}$, $c \setminus F_{xy}$, $x \setminus c^*$, $y \setminus c^*$, $z \setminus c^*$ are nonempty by (F1), (F4) and (F5), and they are pairwise disjoint, because their neurons lie in seven different subsets of $\{F_{xy}, F_{xz}, F_{yz}, c^*\}$.

6 Wheel exclusion

Theorem 6.1. *Each of the 58 codes is minimal (and so has no local obstruction), is not convex, and has a realization without a wheel. In particular, no tuple is a wheel of every realization, and the reading "every realization has a wheel" fails too. Consequently each of the 58 codes is a counterexample to Conjecture 1.1 of [1].*

Proof. Minimality holds by construction (Section 3.4), and non-convexity follows from Lemma 4.2 or Theorem 5.1. Each of the 58 codes is a cone $C = \{\emptyset\} \cup \{c \cup \{i\} : \emptyset \neq c \in B\}$ over a max-intersection-complete code B, which is convex by [4, Theorem 1.2]. Choose any convex realization $\{U_j\}_{j \neq i}$ of B and define $U_i := \bigcup_{j \neq i} U_j$. The code of this family is C: a point lying in some U_j has codeword $\{i\} \cup c_B(x)$ with $c_B(x) \neq \emptyset$, and a point outside every U_j has codeword $\emptyset$. So U is a realization of C by open sets in which only U_i fails to be convex, and U_φ is convex whenever $\varphi \neq \{i\}$, because $i \in \varphi$ implies $U_\varphi = U_{\varphi \setminus \{i\}}$.

A tuple that fails P(i) or P(ii) fails W(i) or W(ii) ([11, Proposition 4.3]), so it is not a wheel. A tuple whose rim is $\tau = \{i\}$ never satisfies P(ii): every nonempty codeword contains i, so $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \{i\}$ is a face as soon as $\sigma_1 \cup \sigma_2 \cup \sigma_3$ is. Every other candidate, that is, every tuple satisfying P(i) and P(ii) with $\tau \neq \{i\}$, also has $\tau \neq \emptyset$, because for $\tau = \emptyset$ condition P(ii) says $\sigma_1 \cup \sigma_2 \cup \sigma_3 \notin \Delta(C)$, which contradicts P(i). Hence neither τ nor any $\sigma_j \cup \tau$ equals $\{i\}$, so U_τ and every $U_{\sigma_j} \cap U_\tau = U_{\sigma_j \cup \tau}$ are convex. The hypothesis of W(iii) is therefore satisfied. Every tuple that fails P(iii)$^\circ$ has some $U_{\sigma_j \cup \tau}$ empty. Every partial-wheel tuple has a face ρ with $\mathrm{Tk}(\sigma_1 \cup \tau) \cup \mathrm{Tk}(\sigma_3 \cup \tau) \subseteq \mathrm{Tk}(\rho)$ and $\mathrm{Tk}(\sigma_2 \cup \tau) \cap \mathrm{Tk}(\rho) = \emptyset$ (such a ρ is never $\{i\}$, whose trunk meets every nonempty trunk). By

Lemma 2.1, every segment from $U_{\sigma_1\cup\tau}$ to $U_{\sigma_3\cup\tau}$ then stays in the convex set U_ρ, which is disjoint from $U_{\sigma_2\cup\tau}$. □

As a separate check, exhaustive searches over all ordered tuples $(\sigma_1,\sigma_2,\sigma_3,\tau)\in\Delta(C)^4$ (as in [11, Definition 4.2], with empty faces allowed) found no sprocket, wheel frame or wire wheel in any of the 58 codes, and no partial wheel of any kind in the 18 (Section 10). The 40 do have partial wheels (the seven-neuron code has 1,152 such tuples, or 192 up to permutation of the spokes), and each of these fails W(iii) in the realization constructed above. Each search gives the correct answer on a published example: the sprocket search on C_{24} of [1, Example 2.19] (the code of Section 7), the wheel-frame search on $C^\star=\{2345,123,134,145,13,14,23,34,45,3,4,\emptyset\}$ of [11, Example 2.7] and on the code $\{12,13,23,1,2,\emptyset\}$ of [11, Example 4.17], and the wire-wheel search on $C_{\mathrm{TL}}=\{123,145,245,246,346,24,45,46,1,2,3,\emptyset\}$ ([11, Example 4.7 and Proposition 4.13]), which has the wire wheel $(1,2,3,4)$ and no sprocket.

Remark. The non-minimal L24 code $D_{\max}$ of Section 7 is non-convex, yet no tuple is a wheel of every realization. A spoke that is not a face makes W(i) fail, a rim τ that is not a face gives $U_\tau=\emptyset$ and makes W(iii) fail, and a tuple of faces that fails P(i) or P(ii) fails W(i) or W(ii) in every realization ([11, Proposition 4.3]). Each of the 18,168 tuples of faces of $D_{\max}$ that satisfy P(i) and P(ii) fails P(iii)°. So some $U_{\sigma_j\cup\tau}$ is empty in every realization, and W(iii) fails in every realization in which each U_φ with $\varphi\supseteq\tau$ is convex. Such a realization exists for every nonempty face τ. Realize the code $\{c\setminus\tau : c\in\mathrm{Tk}(\tau)\}$ by convex open sets V_j ($j\notin\tau$) inside an open rectangle $X\subseteq\mathbb{R}^2$, chosen so that they cover X exactly when $\tau\notin D_{\max}$. Put $U_j=X$ for $j\in\tau$ and $U_j=V_j$ otherwise. Then add, far from X and from each other, one open square for each nonempty codeword c with $\tau\not\subseteq c$, placed in U_j for every $j\in c$. The resulting code is $D_{\max}$, and every U_φ with $\varphi\supseteq\tau$ equals $X\cap\bigcap_{j\in\varphi\setminus\tau}V_j$, which is convex. Sets V_j of this kind exist for all 27 nonempty faces, and $\tau=\{1\}$, where $\{2,3,4,5,6,23,26,45,46,246\}$ is realized in $X=(0,10)\times(0,3)$ by $V_3=\{x<2\}$, $V_2=\{x>1,\ x+y<6\}$, $V_6=\{3<x<7\}$, $V_4=\{x<9,\ x-y>3\}$ and $V_5=\{x>8\}$, each intersected with X. On the other hand, each singleton is the rim of some tuple satisfying P(i) and P(ii). Since every realization by open sets contains a non-convex U_j, such a tuple is a wheel of that realization, and so $D_{\max}$ does have a wheel. The 58 codes avoid this because the apex singleton can never serve as a rim. For this reason, the report presents the 58 as counterexamples.

7 Split-cover pinch

Example 3.22 of [1] is the L24 code $C_{24}=\{123,1246,145,356,12,14,3,5,6,\emptyset\}$, which is non-convex because it has a sprocket. That paper left open which codes D with $C_{24}\subseteq D\subseteq\Delta(C_{24})$ are convex. Since $\Delta(C_{24})$ has 28 faces and C_{24} has 10 codewords, there are $2^{18}=262{,}144$ such codes. Four of the free faces act as *witnesses*: exact convex realizations, checked by listing every cell of the arrangement in arithmetic, show that $C_{24}\cup\{1\}$, $C_{24}\cup\{124\}$, $C_{24}\cup\{56\}$ and $C_{24}\cup\{36\}$ are convex, and by [4, Theorem 1.3] every D containing one of them is convex as well. The other $2^{14}=16{,}384$ codes are exactly the subsets of a single maximal code $D_{\max}=\Delta(C_{24})\setminus\{1,36,56,124\}$. All of them are non-convex, by the criterion below applied to $D_{\max}$ combined with [4, Theorem 1.3].

Lemma 7.1 (pinch lemma)**.** *Let $A,B,C\subseteq\mathbb{R}^d$ be open convex sets. Assume $p\in A$, $q\in B$, $v\in A\cap B\cap C$, and that some point $m=(1-\lambda)p+\lambda q$ with $\lambda\in(0,1)$ satisfies $m\in C$, $m\notin A$, $m\notin B$. Then $A\cap C\neq B\cap C$.*

Figure 7 shows the configuration.

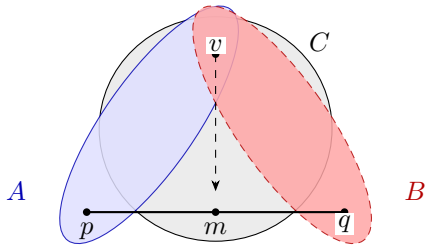


Figure 7: The pinch lemma. The sets A and B overlap at the top, where the point $v \in A \cap B \cap C$ lies. The points $p \in A$ and $q \in B$ sit in the two legs, and the point m of the segment between them belongs to C but to neither A nor B. If $A \cap C$ and $B \cap C$ were one and the same set X, a point pushed from v toward m (dashed) would remain in X all the way to m (Steps 2–3), and a hyperplane separating m from B would then contradict $X \subseteq B$ (Step 4).

Proof. Assume for contradiction that $X := A \cap C = B \cap C$. Then X is open and convex, and $v \in X$.

Step 0. First, $p \notin C$. Otherwise $p \in A \cap C = X \subseteq B$, and since $q \in B$ as well, convexity gives $m \in [p, q] \subseteq B$. By symmetry, $q \notin C$.

Step 1. For $t \in [0, 1]$ let $y_t = v + t(m - v)$, so that $y_t \in C$. The set $T = \{t \in [0, 1] : y_t \in X\}$ is convex and relatively open in $[0, 1]$, and it contains 0 but not 1. Hence $T = [0, \tau)$ for some $\tau \in (0, 1]$.

Step 2 (push). Fix $t < \tau$, write $y = y_t \in X \subseteq A \cap B$, and choose $\rho > 0$ with $\mathbb{B}(y, \rho) \subseteq C$. Let $e_p = (p - y)/|p - y|$ and $e_q = (q - y)/|q - y|$, which are well defined by Step 0. For $0 < s < \min(\rho, |p - y|)$ the point $y + se_p$ lies on $[y, p] \subseteq A$ and also in C, so it lies in X. In the same way $y + se_q \in X$ for $0 < s < \min(\rho, |q - y|)$. Next, $m - y = \alpha e_p + \beta e_q$ where $\alpha = (1 - \lambda)|p - y| > 0$ and $\beta = \lambda|q - y| > 0$. So for every $\mu > 0$ with $\mu(\alpha + \beta) < \min(\rho, |p - y|, |q - y|)$,

$$y + \mu(m - y) = \frac{\alpha}{\alpha + \beta}\big[y + \mu(\alpha + \beta)e_p\big] + \frac{\beta}{\alpha + \beta}\big[y + \mu(\alpha + \beta)e_q\big] \ \in X.$$

Because $y_t + \mu(m - y_t) = y_{t+\mu(1-t)}$, it follows that $t + \mu(1 - t) \in T$.

Step 3 ($\tau < 1$ cannot occur). Since C is open and $y_\tau \in C$, there is ρ_0 with $\mathbb{B}(y_\tau, 2\rho_0) \subseteq C$, and then $\mathbb{B}(y_t, \rho_0) \subseteq C$ for all t close to τ. By Step 0, moreover, $|p - y_t| \to |p - y_\tau| > 0$ and $|q - y_t| \to |q - y_\tau| > 0$, and $\alpha + \beta = (1 - \lambda)|p - y_t| + \lambda|q - y_t| < R := |p - y_\tau| + |q - y_\tau| + 1$ for t close to τ. A single value $\mu_0 := \min(\rho_0, \inf|p - y_t|, \inf|q - y_t|)/R > 0$, with the infima taken over t close to τ, therefore serves for all such t, which gives $t + \mu_0(1 - t) < \tau$. Letting $t \to \tau^-$ gives $\mu_0(1 - \tau) \le 0$, and so $\tau = 1$.

Step 4 ($\tau = 1$ cannot occur). In this case $y_t \in X$ for every $t < 1$, and $y_t \to m$. Since $m \notin B$ and B is open and convex, there is an affine f with $f > 0$ on B and $f(m) \le 0$. Combining $f(m) = (1 - \lambda)f(p) + \lambda f(q)$ with $f(q) > 0$ gives $f(p) < f(m) \le 0$, and in particular $p \ne m$. Let $u = (p - m)/|p - m|$ and $s = \min(\rho_0, |p - m|/2)$, where $\mathbb{B}(m, 2\rho_0) \subseteq C$, and for t close to 1 define $z_t = y_t + s\,(p - y_t)/|p - y_t|$. Then $z_t \in A$ because it lies on $[y_t, p]$, and $z_t \in C$ because $|z_t - y_t| = s \le \rho_0$. Thus $z_t \in X \subseteq B$, and so $f(z_t) > 0$. On the other hand $z_t \to m + su$ and $f(m + su) = f(m) + s\,(f(p) - f(m))/|p - m| < 0$, so $f(z_t) < 0$ for t close to 1. This is a contradiction. □

Definition 7.2 (split-cover certificate)**.** A split-cover certificate is a tuple $(r; a, b, e; J \mid K; \sigma, \tau)$, with $r, a, b, e \in [n]$ distinct, that satisfies the conditions below, where $M := \{j \ne r : \mathrm{Tk}(\{r, j\}) \ne \emptyset\}$.

(SC0) $\{r\} \notin C$.

(SC1) $\mathrm{Tk}(\{a,e\}) \neq \emptyset$ and every codeword in it contains b; $\mathrm{Tk}(\{b,e\}) \neq \emptyset$ and every codeword in it contains a.

(SC2) $\mathrm{Tk}(\{a,b,e\}) \neq \emptyset$.

(SC3) $M = J \sqcup K$ with $a \in J$, $b \in K$.

(SC4) $\sigma \in C$ with $\{r,a\} \subseteq \sigma$ and $\sigma \cap K = \emptyset$.

(SC5) $\tau \in C$ with $\{r,b\} \subseteq \tau$ and $\tau \cap J = \emptyset$.

(SC6) For every $j \in J$, $k \in K$, every codeword containing $\{r,j,k\}$ contains e and omits both a and b.

Theorem 7.3 (split-cover pinch)**.** *A code C that admits a split-cover certificate is not convex.*

Proof. Let U be a convex realization and fix a certificate. (1) By (SC0), every codeword $c \ni r$ contains some $j \neq r$. Then $c \in \mathrm{Tk}(\{r,j\})$, so $j \in M$ and $c \in \mathrm{Tk}(\{j\})$. This shows $\mathrm{Tk}(\{r\}) \subseteq \bigcup_{j \in M} \mathrm{Tk}(\{j\})$, and Lemma 2.1 gives $U_r \subseteq \bigcup_{j \in M} U_j$. (2) Using (SC4) and (SC5), choose p in the atom of σ and q in the atom of τ. Then $p, q \in U_r$, $p \in U_a$, $q \in U_b$, $p \notin U_k$ for $k \in K$, and $q \notin U_j$ for $j \in J$. As U_r is convex, $[p,q] \subseteq U_r \subseteq (\bigcup_J U_j) \cup (\bigcup_K U_k)$. The sets $[p,q] \cap \bigcup_J U_j$ and $[p,q] \cap \bigcup_K U_k$ are relatively open in $[p,q]$. The first contains p but not q, the second contains q but not p, and together they cover the connected segment. They must therefore meet at some $m = (1-\lambda)p + \lambda q$, $\lambda \in (0,1)$, with $m \in U_j \cap U_k$ for some $j \in J$, $k \in K$. (3) Since $m \in U_r \cap U_j \cap U_k$, we have $c(m) \supseteq \{r,j,k\}$, and (SC6) gives $m \in U_e$, $m \notin U_a$, $m \notin U_b$. (4) By (SC2) there is a point $v \in U_a \cap U_b \cap U_e$. Lemma 7.1, applied to U_a, U_b, U_e in place of A, B, C, now gives $U_a \cap U_e \neq U_b \cap U_e$. (5) Condition (SC1) states that $\mathrm{Tk}(\{a,e\}) \subseteq \mathrm{Tk}(\{b\})$ and $\mathrm{Tk}(\{b,e\}) \subseteq \mathrm{Tk}(\{a\})$. By Lemma 2.1, $U_a \cap U_e \subseteq U_b$ and $U_b \cap U_e \subseteq U_a$, so $U_a \cap U_e = U_b \cap U_e$, which contradicts (4). □

The code

$$D_{\max} = \{\emptyset, 2, 3, 4, 5, 6, 12, 13, 14, 15, 16, 23, 24, 26, 35, 45, 46, 123, 126, 145, 146, 246, 356, 1246\}$$

admits, for example, the certificate $(1; 3, 5, 6; \{2,3\} \mid \{4,5,6\}; 123, 15)$. There are exactly 16 certificates, all with $r = 1$, $e = 6$ and $\{a,b\} = \{3,5\}$, and four up to the choice of σ and τ. The code $D_{\max}$ has no local obstruction (the link of every missing face is collapsible) and no partial wheel. This gives the following classification.

Theorem 7.4. *Among the 262,144 codes D with $C_{24} \subseteq D \subseteq \Delta(C_{24})$, exactly 245,760 are convex, namely those that contain at least one of $\{1\}, \{36\}, \{56\}, \{124\}$. The remaining 16,384 are not convex.*

Proof. A code D that contains one of the four witness codes W satisfies $W \subseteq D \subseteq \Delta(C_{24}) = \Delta(W)$, so it is convex by [4, Theorem 1.3]. There are $2^{18} - 2^{14} = 245{,}760$ such codes. Each of the other 2^{14} codes satisfies $C_{24} \subseteq D \subseteq D_{\max}$ and $\Delta(D) = \Delta(C_{24})$. If such a D were convex, then $D_{\max}$, which contains it, would be convex by [4, Theorem 1.3], contradicting Theorem 7.3. □

The search for certificates finds none for the four witness codes or for 3,000 randomly generated max-intersection-complete codes. It does find certificates for the five codes $C_{24} \cup \{13, 35\}$, $C_{24} \cup \{15, 35\}$, $C_{24} \cup \{16, 35\}$, $C_{24} \cup \{13, 15, 35\}$ and $C_{24} \cup \{15, 16, 35\}$ of the family, which had earlier been shown non-convex by separate arguments (all five are subsets of $D_{\max}$), and for all 16,384 subsets of $D_{\max}$.

8 Published unknowns

Lemma 4.1 gives a rule within the framework of Jeffs, Lienkaemper and Youngs [9].

Lemma 8.1 (absorption rule)**.** *Let U be a convex realization of C, and let $v_0, \dots, v_k$ be the simple sequence of codewords met along a segment, after loops are erased. Then consecutive codewords are comparable, $v_i \cap v_j \subseteq v_m$ for every $0 \le i < m < j \le k$, and the following* absorption rule *holds: if neuron $\alpha \in v_i$, neuron $\beta \in v_j$, $\mu \subseteq v_m$ with $i \le m \le j$ and $\mathrm{Tk}(\mu \cup \{\alpha\}) = \mathrm{Tk}(\mu \cup \{\beta\}) \neq \emptyset$, then $\alpha, \beta \in v_m$. Consequently, if some pair of codewords of C admits no sequence with these three properties, then C is not convex.*

Proof. Comparability is [9, Lemma 2.1]: each open convex U_j meets the segment in an open interval, so consecutive distinct codewords are comparable, and loop erasure keeps consecutive codewords adjacent. The second property follows from the convexity of each U_j, and every subsequence obtains it, in particular the loop-erased one. For the rule, apply Lemma 4.1 with $A = U_\alpha$, $B = U_\beta$ and $M = U_\mu$, so that $A \cap M = B \cap M \neq \emptyset$ by Lemma 2.1. The point of the segment in the atom of v_m lies between points of A and B, hence in $\mathrm{conv}(A \cup B) \cap M = A \cap M \subseteq A \cap B$. Since a convex realization produces such a sequence for any two codewords, the final statement follows. □

For reduced indecomposable six-neuron codes with four to seven facets, the census of [11] records which codes are max-intersection-complete, which have a wheel, and which are undecided; 1,800 fall in the last group. The criterion flags **63** of these 1,800 codes (3 with five facets, 18 with six, 42 with seven). The original search bounded the size of μ and capped the number of search steps for each pair of codewords. The search was completed in two further runs, in which the codes that had hit the cap were searched again with a larger bound on μ. When these runs finished, no code was left undecided. Two later implementations allow every $\mu \subseteq v_m$ and impose no step cap. Both process all 1,800 codes and flag exactly the same 63. The same two implementations were run on two controls. They flag none of the 1,391 max-intersection-complete codes of the census, all of which are convex. They flag 315 of the 316 codes that the census proves non-convex by wheels, including 312 of the 313 with five to seven facets. The one code they miss, `FiveFac649`, is a wheel-frame code with 20 codewords that admits a sequence with all three properties for every pair of codewords. The criterion is therefore sufficient for non-convexity but not complete on six neurons. Appendix A lists the 63 identifiers in the census's own numbering, and Table 2 gives a summary.

Table 2: The absorption criterion applied to the published six-neuron census of [11] (reduced, indecomposable codes). "MIC" means max-intersection-complete, hence convex; "wheel" means shown non-convex by a sprocket, wheel frame or wire wheel; "unknown" means left undecided by the census.

facets	MIC	wheel	unknown	flagged by absorption	share of unknown
4	4	3	0	–	–
5	79	36	96	3	3.1%
6	399	118	535	18	3.4%
7	909	159	1,169	42	3.6%
all	1,391	316	1,800	63	3.5%

9 Other nerve classes

Let $k(C)$ denote the number of pairs $\{i, j\}$ for which $F_i \cap F_j$ is contained in a third facet.

Theorem 9.1. *An L27 code, whether minimal or not and on any number of neurons, is convex iff it has no local obstruction.*

Proof. Every intersection of two or three facets is a mandatory face (by the nerve lemma its link is homotopy equivalent to S^0 or S^1), so a code without local obstructions is max-intersection-complete and hence convex by [4, Theorem 1.2]. The converse is [1, Lemma 2.23]. □

Theorem 9.2. *A minimal L25 or L26 code C is non-convex iff $k(C) = 1$, and in that case it has a sprocket.*

Proof. Suppose first that $k = 1$. Writing $F_a \cap F_b \subseteq F_c$ for the containment and F_d for the fourth facet, the tuple $(F_a \cap F_d,\ F_c \cap F_d,\ F_b \cap F_d;\ F_a \cap F_b)$ with witnesses $F_a \cap F_c$, $F_b \cap F_c$ satisfies P(i), P(ii), P(iii)° and S(1)–S(3) of [1, Definition 2.16]. When $k = 0$, every max-intersection face is mandatory and the code is convex by the same reasoning as for L27. When $k \geq 2$, type reduction (Theorem 3.3) leaves finitely many explicit codes on at most 11 neurons. The 96 of these that are not already max-intersection-complete are convex by results of [11]. After redundant neurons are deleted ([11, Proposition 6.3]), 76 of them are codes on five neurons with no local obstruction and no wheel frame, and these are convex by [11, Theorem 3.8]. The other 20 are decomposable codes on six neurons ([11, Definition 6.4]) whose embedded code is a single neuron and whose ambient code is either max-intersection-complete or a code on five neurons with no local obstruction and no wheel frame; these are convex by [11, Theorem 6.6]. □

After completing this work, I learned from an author of [1] that their group had independently settled the minimal L25, L26 and L27 cases, using the same argument for L27. I therefore do not claim those three results as new.

10 Verification

- **Enumeration.** The 606 complexes, the 353/253 split and the 115/80/58 decision were all reproduced from scratch by a second implementation developed independently of the first. In every case contractibility was decided with a certificate in both directions: an explicit sequence of elementary collapses shows that a complex is contractible, and nonvanishing reduced homology shows that it is not. As a control, the checker correctly declines to call contractible a six-vertex triangulation of the projective plane, which is acyclic over $\mathbb{Q}$ but not contractible.

- **Certificates.** The Lemma 4.2 certificates (72 ordered tuples, one per code up to the symmetry of (Z1)–(Z5)), the front-obstruction certificates for the 40 and the split-cover certificates (16 on $D_{\max}$, or 4 up to the choice of σ and τ) were each recomputed by an independent search. The front-obstruction search found no certificate on any of 2,997 codes known to be convex. These are the 2,500 max-intersection-complete records in the data files of the census of [11], namely the 1,391 codes in the MIC column of Table 2 together with the 1,109 that the census sets aside as reducible or decomposable; the 115 convex codes of Section 3.4; 320 codes of exact box arrangements in $\mathbb{R}^2$ and $\mathbb{R}^3$; the 58 coned codes; and four named codes, namely $C_{24} \cup \{1\}$, the seven-codeword max-intersection-complete code $\{123, 134, 145, 13, 14, 1, \emptyset\}$, B_0 and $C_0 \cup \{\{i\}\}$. The split-cover search finds nothing on the four witness codes or on 3,000 randomly generated max-intersection-complete codes, and the Lemma Z search finds nothing on the 58 coned codes.

- **Wheels.** The absence of sprockets, wheel frames and wire wheels in the 58, and of partial wheels in the 18, was recomputed by an implementation transcribed from [11, Definitions 4.2,

4.4, 4.8, 4.14], using the paper's own examples as positive controls (C_{24} for sprockets, $C^\star$ and Example 4.17 for wheel frames, and C_{TL} of [11, Example 4.7] for wire wheels). For all 58 codes this was a census over the ordered tuples $(\sigma_1, \sigma_2, \sigma_3, \tau) \in \Delta(C)^4$, with the empty face allowed as spoke, rim and witness. In the 18, 2,432,797,440 tuples satisfy P(i) and P(ii); in the 40, 35,620,096,512 do, of which 138,240 are partial wheels and none is a sprocket. Wire wheels and wheel frames were enumerated directly from their definitions, and none exists in any of the 58.

- **Realizations.** The codes of the four witness realizations, the $\mathbb{R}^3$ realization of B_0, and the 27 realizations of $D_{\max}$ in the remark of Section 6 were recomputed by exact arrangement-cell enumeration. The union realizations of Section 6 were checked through their induced codewords.

- **Proof review.** Each of the four principal hand proofs, those of Lemmas 4.1 and 7.1 and Theorems 5.1 and 7.3, was checked line by line at least twice.

11 Limitations

No professional mathematician has yet read Step B of Theorem 5.1 in full. The instructor has examined the smallest of the 18 codes and confirmed it after an initial reading, but at the time of submission he had not read the rest of Sections 4–7. The 63 flagged census codes are established by certificate, and no person has checked them one by one.

The wheel status of the 63 flagged census codes under the flexible reading is not determined here, and the other 1,737 undecided census codes remain unresolved. The open embedding dimensions of the convex cones $C \cup \{\{i\}\}$ of the 58 counterexample codes are not computed here, except for $C_0 \cup \{\{i\}\}$ as discussed in Section 5.

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A Census identifiers

Identifiers are taken from the `id` field of the files `FiveFac_WIRE_WHEEL_6.sage`, `SixFac_WIRE_WHEEL_6.sage` and `SevenFac_WIRE_WHEEL_6.sage` published with [11].

FiveFac (3): 499, 505, 511.

SixFac (18): 584, 585, 715, 734, 888, 1091, 1118, 1203, 1216, 1224, 1240, 1241, 1255, 1256, 1260, 1261, 1263, 1422.

SevenFac (42): 576, 679, 888, 943, 982, 1008, 1237, 1239, 1243, 1245, 1246, 1251, 1253, 1377, 1694, 1695, 1785, 1921, 1994, 2039, 2040, 2044, 2060, 2093, 2099, 2108, 2112, 2116, 2123, 2136, 2142, 2145, 2148, 2149, 2150, 2151, 2155, 2177, 2194, 2207, 2397, 2403.

B Counterexample codes

The 18 codes of the first shape and the 40 codes of the second shape are given as plain text in the supplementary files `SEND_18_codes.txt` and `SEND_40_codes.txt`. In the first file, neurons are named by their type and the apex is called i. In the second, neurons are numbered $1, \ldots, n-1$, the apex is 0, and the type of a neuron is the set of maximal codewords containing it. Each code has 12 or 14 codewords on 7 to 14 neurons.