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Research Report

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Title of Research Report

On Open Borel Orbits in Conormal Bundles to Schubert Cells

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ON OPEN BOREL ORBITS IN CONORMAL BUNDLES TO SCHUBERT CELLS

DEREK AOKI

ABSTRACT. Let $C(v)$ denote a Schubert cell in the flag variety GL_n/B, and let $X(w)$ be a Schubert variety containing $C(v)$. Mellit has conjectured that, if $X(w)$ is smooth, the existence of an open B-orbit in the conormal bundle to $C(v)$ in $X(w)$ is equivalent to $C(w_0w)$ being contained in the smooth locus of $X(w_0v)$. We prove a stronger version of this conjecture, requiring only that $X(w)$ is factorial. We prove our result by constructing a sequence of graphs associated to the pair v, w such that the existence of a cycle in one of these graphs is equivalent to the non-existence of such an open orbit.

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Declaration of Generative AI Use

During the writing of this paper, the author used ChatGPT 5.6 Sol (OpenAI) for proofreading and copyediting. Additionally, ChatGPT suggested the use of Theorem 5.6, upon being asked whether there were existing results of the form "$|E_1(w_0w, w_0v)| < |E_1(v, w)|$ implies $X(w_0v)$ is singular at e_{w_0w}". Apart from this suggestion, all mathematical content and writing is the author's own.

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Date: September 6, 2026

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1. Introduction

Let $G = \mathrm{GL}_n$ and let $B \subset G$ be its Borel subgroup of upper triangular matrices. The flag variety G/B is stratified by Schubert varieties $X(w)$ defined as the closures of Schubert cells $C(w)$, the orbits of B on G/B acting by left multiplication. The geometry of this stratification has proven fundamental to both representation theory and algebraic combinatorics. Kazhdan and Lusztig conjectured [18, 19] that certain polynomials, indexed by pairs $v, w \in S_n$ such that $v \preceq w$, control the combinatorics of the BGG category $\mathcal{O}$ for Lie(G). These conjectures were later proven independently by Brylinski–Kashiwara [6] and Beilinson–Bernstein [2], who both tied the Kazhdan–Lusztig polynomials $P_{v,w}(q)$ to the geometry of Schubert varieties. For example, the (rational) smoothness of $C(v)$ in $X(w)$ is equivalent to the triviality of $P_{v,w}(q)$ [8].

Motivated by these connections as well as the combinatorial properties of flag varieties, the last few decades have seen interest in understanding the smooth locus of the Schubert variety $X(w)$. The seminal result on this topic by Lakshmibai–Sandhya [20] showed that $X(w)$ is smooth if and only if w avoids 3412 and 4231. Following their result, four independently published papers gave a combinatorial characterization of the pairs $v, w \in S_n$ such that $C(v)$ is contained in the singular locus of $X(w)$ [3, 7, 17, 26]. Lapid–Mínguez showed (albeit in the equivalent language of G-orbits in $G/B \times G/B$) using combinatorial methods that, if w avoids 231, then an open B-orbit in the conormal bundle to $C(v) \subseteq X(w)$ exists if and only if $C(w_0 w)$ is contained in the smooth locus of $X(w_0 v)$ [23]. This open orbit condition originates with the work of Geiß-Leclerc-Schröer [10, 11, 12] and subsequent work on rigid modules over preprojective algebras and related dense orbit conditions [1, 13, 16, 22, 24]. This result of Lapid–Mínguez led Mellit to conjecture that this equivalence in fact holds when $X(w)$ is smooth, which is a strictly weaker condition than w avoiding 231 [25, Conjecture 6.2]. In this paper, we study the following equivalent formulation of Mellit's conjecture.

Conjecture 1.1. *Let $v \preceq w$ such that $X(w)$ is smooth. Then $X(w_0 v)$ is singular at $e_{w_0 w}$ if and only if there is no open B-orbit in the conormal bundle to $C(v) \subseteq X(w)$.*

Our main contribution is the following theorem, which proves a stronger version of the conjecture above.

Theorem 1.2. *Let $v \preceq w$ such that $X(w)$ is factorial. Then $X(w_0 v)$ is singular at $e_{w_0 w}$ if and only if there is no open B-orbit in the conormal bundle to $C(v) \subseteq X(w)$.*

To prove our main theorem, we establish a combinatorial condition equivalent to the existence of an open B-orbit in the conormal bundle. We construct a sequence of graphs $\mathcal{G}_i(v, w)$ that we associate to the pair v, w, which we build from the pairs (a, b) such that $v \prec v s_{ab} \preceq w$. A construction equivalent to $\mathcal{G}_1(v, w)$ had previously been considered by Tsukerman–Williams in [27], called the atom graph. On the other hand, to the best of our knowledge, the higher graphs $\mathcal{G}_i(v, w)$ for $i > 1$ have not been studied. We will show that the presence of a cycle in these graphs is equivalent to the failure of the conormal bundle to contain an open B-orbit. Our approach makes heavy use of the characterization of maximal singular points by Billey–Warrington [3] and their corresponding Bruhat pictures.

In Section 2, we recall a linear algebraic characterization of the existence of an open B-orbit from [23] and a pictorial characterization of the maximal singular points from [3] in terms of Bruhat pictures. In Section 3, we prove that the existence of an open B-orbit implies that $C(w_0w)$ is contained in the smooth locus of $X(w_0v)$, without any assumptions on w. In Section 4, we define the graphs $\mathcal{G}_i(v,w)$ and use them to give a necessary condition for the non-existence of an open B-orbit.

In Section 5, we prove our main theorem. We first prove the following theorem, which is of independent interest because it gives a purely combinatorial characterization of the existence of an open B-orbit.

Theorem 1.3. *There exists an open B-orbit in the conormal bundle to $C(v) \subseteq X(w)$ if and only if each of the graphs $\mathcal{G}_i(v,w)$ is a forest.*

In fact, an algorithm utilizing the above theorem for determining the existence of an open B-orbit runs in $O(n^2)$, whereas the naive algorithm using the linear algebraic characterization of [23] runs in $O(n^6)$. Further, we have that a cycle in $\mathcal{G}_i(v,w)$ for $i > 1$ implies $X(w_0v)$ is singular at e_{w_0w}. We conclude by proving that if $X(w)$ is factorial and $\mathcal{G}_1(v,w)$ contains a cycle then $X(w_0v)$ is singular at e_{w_0w}, using results of Deodhar [8] and Dyer [9] on the nontriviality of Kazhdan-Lusztig polynomials and results of Bousquet–Mélou–Butler [5] and Tsukerman–Williams [27] on the graphs $\mathcal{G}_1(v,w)$.

Remark 1.4. During the preparation of this paper, the preprint [22] by Lapid appeared. In its treatment of type A, it makes use of the same linear algebraic condition from [23] we do, and defines a graph which is the same as $\mathcal{G}_1(v,w)$ up to choice of conventions. This paper also characterizes the existence of an open B-orbit in terms of the termination of a greedy algorithm. Our approach gives a more explicit combinatorial description in terms of the graphs $\mathcal{G}_i(v,w)$. Moreover, our paper handles the case of arbitrary pairs $v \preceq w$, whereas [22] only handles the case $w = w_0$. Because of this, we are able to prove Mellit's conjecture, whereas the goals of [22] are different.

2. Preliminaries

2.1. **Basic Definitions.** Throughout this paper, we work over $\mathbb{C}$. Again, let $G = \mathrm{GL}_n$, and let B be the Borel subgroup of upper triangular elements of G. We write $T \subset B$ for the maximal torus of diagonal matrices. The flag variety G/B is a disjoint union of B-orbits $C(w) = BwB/B$ in G/B called *Schubert cells*, which are indexed by elements w of the Weyl group $W = N(T)/T$. We have an isomorphism $W \cong S_n$, where S_n is the symmetric group on n elements. We write s_{ij} for the element of $W \cong S_n$ transposing i and j. We call the Zariski closure $X(w) = \overline{C(w)}$ a *Schubert variety*. Since each $X(w)$ is stable under the action of B, it is a union of Schubert cells. We define a partial order, called the *Bruhat order*, on $W \cong S_n$ such that $v \preceq w$ if and only if $C(v) \subseteq X(w)$. There exists a unique element denoted $w_0 \in W$, defined by $i \mapsto n+1-i$, which is maximal with respect to the Bruhat order. In the sequel, we refer to permutations by their one line notations. That is, we write $w = [a_1, \ldots, a_n]$ for the permutation w such that $w(i) = a_i$. For instance, we have $w_0 = [n, n-1, \ldots, 1]$.

For $v \in S_n$, we write e_v for the point $vB/B \in G/B$. Writing $B^v = vBv^{-1}$, we define $B_v = B \cap B^v$ for the stabilizer of $e_v \in G/B$ under the B-action. We also write $\mathfrak{b}^v = \mathrm{Lie}(B^v)$ and $\mathfrak{b}_v = \mathrm{Lie}(B_v)$. We identify the tangent space $T_{e_v}(G/B)$ to G/B at e_v with the quotient $\mathfrak{g}/\mathfrak{b}^v$. Writing B^- for the Borel subgroup opposite to B, and writing $\mathfrak{n}_v^-$ for the nilpotent radical of $\mathrm{Lie}((vB^-v^{-1})$, we identify $\mathfrak{g}/\mathfrak{b}^v$ with $\mathfrak{n}_v^-$ as vector spaces. The cotangent space $T^\vee_{e_v}(G/B)$ is the linear dual to $T_{e_v}(G/B)$, and B_v acts naturally on $T_{e_v}(G/B)$. The nondegenerate bilinear form $(X, Y) \mapsto \mathrm{tr}(XY)$ then yields a B_v-equivariant isomorphism $(\mathfrak{g}/\mathfrak{b}^v)^* \cong \mathfrak{n}_v^+$, the nilpotent radical of $\mathfrak{b}^v$, where B_v acts on $\mathfrak{n}_v^+$ via the adjoint action. Explicitly, we fix this identification so that $T^\vee_{e_v}(G/B)$ is spanned by the elementary matrices $E_{v(i)v(j)}$ for $i < j$.

Let $w \in S_n$ such that $w \succeq v$. The following theorem of Lakshmibai–Seshadri gives an explicit description of the tangent spaces to the Schubert varieties as a subspace of $T_{e_v}(G/B)$ [21, Theorem 1].

Theorem 2.1. *The tangent space $T_{e_v}X(w) \subseteq T_{e_v}(G/B)$ has a basis $\{E_{v(j)v(i)} : i < j, vs_{ij} \preceq w\}$, where s_{ij} denotes the permutation transposing i and j.*

Dualizing, this identification yields a B_v-equivariant surjection $T^\vee_{e_v}(G/B) \twoheadrightarrow T^\vee_{e_v}X(w)$, with kernel $K(v, w)$ spanned by $\{E_{v(i)v(j)} : i < j, vs_{ij} \not\preceq w\}$. Thus, $T^\vee_{e_v}X(w)$ has basis

$$\{e_{v(i)v(j)} : i < j, vs_{ij} \preceq w\},$$

where $E_{v(i)v(j)}$ is the class of $E_{v(i)v(j)}$ modulo $K(v, w)$.

We define the *conormal space* $N^\vee_{X(w),gB}C(v)$ to $C(v) \subseteq X(w)$ at $gB \in C(v)$ as the subspace of $T^\vee_{gB}X(w)$ which annihilates $T_{gB}C(v) \subseteq T_{gB}X(w)$. For $gB = e_v$, $T_{gb}C(v)$ is a B_v-submodule of $T_{gB}X(w)$. We write $N^\vee_{X(w)}C(v)$ for the conormal bundle to $C(v)$ in $X(w)$, whose fiber at gB is $N^\vee_{X(w),gB}C(v)$. By definition, there is a short exact sequence of B_v-modules

$$0 \to N^\vee_{X(w),e_v}C(v) \to T^\vee_{e_v}X(w) \to T^\vee_{e_v}C(v) \to 0.$$

We write

$$R(v, w) = \{(i, j) \in [n]^2 : i < j, \quad v \prec vs_{ij} \preceq w\}.$$

Then $N^\vee_{X(w),e_v}C(v)$ is the B_v-submodule of $T^\vee_{e_v}X(w)$ with basis $\{e_{v(i)v(j)} : (i, j) \in R(v, w)\}$.

Definition 2.2. For $x \in N^\vee_{X(w),e_v}C(v)$, we write $x_{v(i)v(j)} \in \mathbb{C}$ such that

$$x = \sum_{(i,j)\in R(v,w)} x_{v(i)v(j)}e_{v(i)v(j)}.$$

We use the convention that $x_{v(i)v(j)} = 0$ if (i, j) is not in $R(v, w)$. We write $\vec{1}$ for the vector $x \in N^\vee_{X(w),e_v}C(v)$ such that every $x_{v(i)v(j)}$ is 1.

We also write $S(v) = \{(i, j) \in [n]^2 : i \leq j, v(i) \leq v(j)\}$ for the pairs (i, j) which index the basis of $\mathfrak{b}_v \subseteq \mathfrak{b}$. Explicitly, $\mathfrak{b}_v$ is spanned by $\{E_{v(i)v(j)} : (i, j) \in S(v)\}$. When $i < j$, note that $(i, j) \in S(v)$ if and only if $v \preceq vs_{ij}$ [4, Theorem 2.1.5].

With $S(v)$ and $R(v, w)$ established, we introduce a matrix whose rank detects open orbits of B_v in the $N^\vee_{X(w),e_v}C(v)$.

Definition 2.3. Let $x \in N^\vee_{X(w),e_v} C(v)$. We define $M^{v,w}(x)$ as a matrix with rows indexed by $R(v,w)$ and columns indexed by $S(v)$. Its entries are given by

$$M^{v,w}(x)_{(a,b),(r,s)} = \begin{cases} x_{v(s)v(b)} & \text{if } a = r \text{ and } (s,b) \in R(v,w), \\ -x_{v(a)v(r)} & \text{if } s = b \text{ and } (a,r) \in R(v,w), \\ 0 & \text{otherwise.} \end{cases}$$

The following gives a purely linear algebraic characterization of the existence of an open B-orbit in $N^\vee_{X(w)} C(v)$, and was established in different language in [23] (cf. Remark 4.11 and the surrounding material).

Theorem 2.4. *The following are equivalent:*

(1) *The action of B on the conormal bundle $N^\vee_{X(w)} C(v)$ admits an open orbit.*
(2) *There exists some $x \in N^\vee_{X(w),e_v} C(v)$ such that* $\operatorname{rank}(M^{v,w}(x)) = |R(v,w)|$.

Proof. Because B acts transitively on $C(v)$, condition (1) is equivalent to the condition that B_v has an open orbit on $N^\vee_{X(w),e_v} C(v)$. This is equivalent to the condition that

$$[\mathfrak{b}_v, x] = N^\vee_{X(w),e_v} C(v).$$

By direct computation, we have

$$\begin{aligned} [E_{v(r)v(s)}, x] &= \sum_{(a,b)\in R(v,w)} x_{v(a)v(b)} [E_{v(r)v(s)}, e_{v(a)v(b)}] \\ &= \sum_{(a,b)\in R(v,w)} x_{v(a)v(b)} (\delta_{as} e_{v(r)v(b)} - \delta_{br} e_{v(a)v(s)}). \end{aligned}$$

Therefore,

$$\begin{aligned} [E_{v(r)v(s)}, x]_{v(a)v(b)} &= \delta_{ar} x_{v(s)v(b)} - \delta_{sb} x_{v(a)v(r)} \\ &= M^{v,w}(x)_{(a,b),(r,s)}. \end{aligned}$$

In particular, our matrix $M^{v,w}(x)$ has full rank $|R(v,w)|$ if and only if $[\mathfrak{b}_v, x]$ is the full conormal space $N^\vee_{X(w),e_v} C(v)$. □

2.2. **Bruhat Pictures.** Before continuing, we introduce Bruhat pictures and related definitions from [3], which allow us to visualize the set $R(v,w)$ and give a pictorial characterization of maximal singular points. We will then recall several results on Bruhat pictures and from [3].

Definition 2.5. Given $v \preceq w$ in S_n, the Bruhat picture for the pair (v,w) is the n by n grid with nodes marked by filled and unfilled dots. For $i \in [n]$, writing $\mathrm{pt}_v(i)$ for the point $(i, v(i)) \in \mathbb{Z}^2$. We place a filled dot $\bullet$ at $\mathrm{pt}_v(i)$, an unfilled dot $\circ$ at $\mathrm{pt}_w(i)$, and a filled dot overlaid on an unfilled one $\odot$ whenever $w(i) = v(i)$. We denote by $\mathrm{Br}(v,w)$ the Bruhat picture of the pair (v,w).

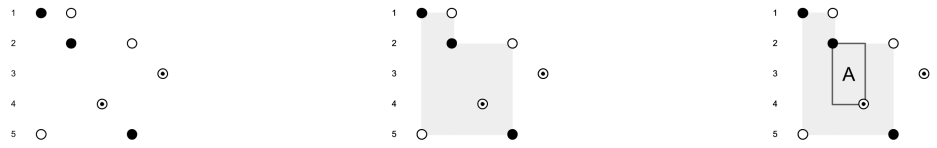
Remark 2.6. It is clear from the definitions of Bruhat pictures and w_0 that we can obtain $\mathrm{Br}(w_0 w, w_0 v)$ by reflecting $\mathrm{Br}(v,w)$ over a vertical axis, replacing the filled dots with unfilled dots, and replacing the unfilled dots with filled dots.

For instance, below in Figure 1a is the plot for $v = [1, 2, 5, 3, 4]$ and $w = [2, 4, 5, 3, 1]$.

In general, we only work with particular subsets of the Bruhat picture, labeling only the rows i and points $\mathrm{pt}_v(i), \mathrm{pt}_w(i)$ of interest.

Definition 2.7. We define the rank function $r_w(p, q) := \#\{i \leq p : w(i) \geq q\}$ for $p, q \in \mathbb{Z}$, and the difference function $d_{v,w}(p, q) := r_w(p, q) - r_v(p, q)$.

In $\mathrm{Br}(v, w)$, we shade the points (p, q) in $\mathbb{Z}^2$ such that $d_{v,w}(p, q)$ is positive. For instance, below in Figure 1b is the plot of the pair v and w in Figure 1a, now with shading.



(A) The Bruhat picture $\mathrm{Br}(v, w)$ for $v = [1, 2, 5, 3, 4]$ and $w = [2, 4, 5, 3, 1]$.

(B) The same Bruhat picture with shading. On the shaded region, we have $d_{v,w} \equiv 1$, and outside it, we have $d_{v,w} \equiv 0$.

(C) The same Bruhat picture with the region $\mathcal{A}_{24}(v)$ indicated as region A.

FIGURE 1. Bruhat pictures for $v = [1, 2, 5, 3, 4]$ and $w = [2, 4, 5, 3, 1]$.

Definition 2.8. For $i < j$, we write $\mathcal{A}_{ij}(v) = (i, j) \times (v(i), v(j)) \subseteq \mathbb{Z}^2$.

We now recall some general results on properties of $d_{v,w}$ from [3].

Lemma 2.9 ([3, Lemma 3, Fact 7])**.** *The following properties of $d_{v,w}$ hold.*

1. *We have $v \preceq w$ if and only if $d_{v,w}(p, q) \geq 0$ for all $p, q \in \mathbb{Z}$.*
2. *Let $v \preceq w$ and $i \leq j$ such that $v \prec vs_{ij}$ (i.e., $v(i) < v(j)$). Then $(i, j) \in R(v, w)$ if and only if $d_{v,w}(p, q)$ is positive for $(p, q) \in \mathcal{A}_{ij}(v)$.*

This lemma allows us read off the set $R(v, w)$ from $\mathrm{Br}(v, w)$. In particular, we have $(i, j) \in R(v, w)$ if and only if the box $\mathcal{A}_{ij}(v)$ (whose top left corner is $\mathrm{pt}_v(i)$ and bottom right corner is $\mathrm{pt}_v(j)$) is shaded. Applying this to Figure 1b, we have by inspection of the Bruhat picture that $v \preceq w$ and that $R(v, w) = \{(1, 2), (2, 4), (2, 5), (4, 5)\}$.

Definition 2.10. Define the set of *maximal singular points* $\mathrm{maxsing}(w)$ to be the set of $v \preceq w$ such that $X(w)$ is singular at e_v, and for any v' with $v \prec v' \preceq w$, $X(w)$ is smooth at $e_{v'}$.

The main theorem of [3] is a characterization of $\mathrm{maxsing}(w)$.

Theorem 2.11 (Billey-Warrington)**.** *We have $v \in \mathrm{maxsing}(w)$ if and only if $v = wc$, with c a cycle of the form $(\alpha_1, \ldots, \alpha_m, \beta_k, \ldots, \beta_1)$, satisfying the following conditions. We emphasize*

that the permutation c here is written in cycle notation, contrary to our prevailing convention. First, we put the following requirements on c.

$$1 \leq \alpha_1 < \cdots < \alpha_m \leq n, \text{ with } w(\alpha_1) > \cdots > w(\alpha_m), \text{ and}$$
$$1 \leq \beta_1 < \cdots < \beta_k \leq n, \text{ with } w(\beta_1) > \cdots > w(\beta_k),$$

Second, one of the following cases must hold.

(1) *The 4231 Case: $k, m \geq 2$, $w(\alpha_m) > w(\beta_1)$, and*
$$\alpha_1 < \beta_1, \ldots, \beta_{k-1} < \alpha_2, \ldots, \alpha_m < \beta_k.$$
Also, in Figure 2a, which depicts a subset of $\mathrm{Br}(v, w)$ in this case, the shaded region contains no filled or unfilled dots.

(2) *The 3412 Case: $k, m \geq 2$*
$$w(\alpha_{m-1}) > w(\beta_1) > w(\alpha_m) > w(\beta_2), \text{ and}$$
$$\beta_1, \ldots, \beta_{k-1} < \alpha_1 < \beta_k < \alpha_2, \ldots, \alpha_m.$$
Also, the shaded region of Figure 2b contains no filled or unfilled dots.

(3) *The 45312 Case: $k = m = 2$,*
$$w(\alpha_1) > w(\beta_1) > w(\alpha_2) > w(\beta_2), \text{ and}$$
$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2.$$
Also, if we have filled dots in the shaded region of Figure 3, then they are concentric circles, and they are in decreasing order as indicated.

While the above cases are somewhat difficult to parse, the pairs v, w can be described in terms of their Bruhat pictures. In the 4231 Case, we have Figure 2a. In the 3412 Case, we have Figure 2b. Finally, in the 45312 Case, we have Figure 3. Importantly, the shaded regions below have no points in them (except for the concentric circles in the 45312 Case). The dotted lines in these figures simply indicate that the pattern repeats.

3. Maximal Singular Points

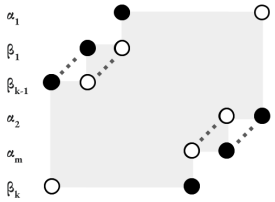
In this section, we prove that if $w_0 w \in \mathrm{maxsing}(w_0 v)$, then there is no open B_v-orbit in $N^{\vee}_{X(w),e_v} C(v)$. We will do this via a directed graph $\mathcal{G}_1(v, w)$. We address the 4231 and 3412 Cases in Section 3.1, and we address the 45312 Case in Section 3.2. In Section 3.2, we use this to conclude that if $X(w_0 v)$ is singular at $e_{w_0 w}$, then there is no open B_v-orbit in $N^{\vee}_{X(w),e_v)} C(v)$.

3.1. Maximal Singular Points in the 4231 and 3412 Cases.

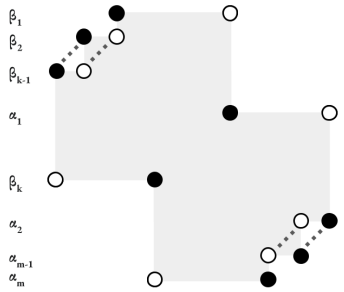
In this subsection, we will prove that for any $v \preceq w$ such that $w_0 w \in \mathrm{maxsing}(w_0 v)$ in the 4231 or 3412 Cases, the rank of $M^{v,w}(x)$ is at most $|R(v, w)| - 1$, and thus that there is no open orbit in $N^{\vee}_{X(w),e_v} C(v)$.

We will now introduce a graph $\mathcal{G}_1(v, w)$ such that a cycle in the graph gives us information about the rank of $M^{v,w}(x)$.

Definition 3.1. Define the partial order $\trianglelefteq$ on $R(v, w)$ such that $(i, j) \trianglelefteq (i', j')$ if and only if we have the containment of rectangles $\mathcal{A}_{ij}(v) \subseteq \mathcal{A}_{i'j'}(v)$.



(A) The Bruhat picture $\mathrm{Br}(v, w)$ with $v \in \mathrm{maxsing}(w)$ in the 4231 Case. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. The shaded region contains no filled or unfilled points. By the dotted lines, we mean a continuation of the staircase pattern.



(B) The Bruhat picture $\mathrm{Br}(v, w)$ with $v \in \mathrm{maxsing}(w)$ in the 3412 Case. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. The shaded region contains no filled or unfilled points. By the dotted lines, we mean a continuation of the staircase pattern.

FIGURE 2

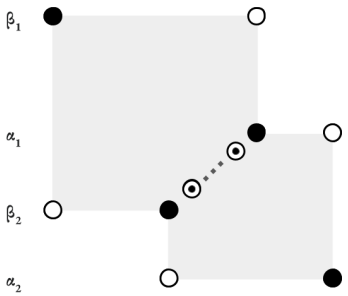


FIGURE 3. The Bruhat picture $\mathrm{Br}(v, w)$ with $v \in \mathrm{maxsing}(w)$ in the 45312 Case. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. The shaded region contains no filled or unfilled points other than the indicated concentric circles. By the dotted line, we mean that the concentric circles continue down and to the left in order. That is, if $\mathrm{pt}_i(v)$ and $\mathrm{pt}_j(v)$ are two of these dots with $i < j$, then $\mathrm{pt}_i(v)$ is to the top right of $\mathrm{pt}_j(v)$.

We let

$$E_0(v, w) = \{(i, i) : (i, j) \in R(v, w) \text{ or } (j, i) \in R(v, w) \text{ for some } j\},$$

and let $E_1(v, w)$ be the elements of $R(v, w)$ which are minimal with respect to $\trianglelefteq$. Visually, (i, j) is in $E_1(v, w)$ if there are no filled dots in the rectangle $\mathcal{A}_{ij}(v)$. We define the directed graph

$\mathcal{G}_1(v,w)$, with vertices labeled by $E_0(v,w)$ and edges labeled by $E_1(v,w)$, such that the source of the edge (i,j) is (i,i) and the target is (j,j). This turns out to be the same, up to a choice of convention, as the atom graph considered by Tsukerman–Williams in [27]. By construction, the graph $\mathcal{G}_1(v,w)$ has no self-edges or multi-edges.

For example, in our previous example Figure 1b, we have $(2,4) \trianglelefteq (2,5)$ and $(4,5) \trianglelefteq (2,5)$. Thus, we have $E_0(v,w) = \{(1,1),(2,2),(4,4),(5,5)\}$ and $E_1(v,w) = \{(1,2),(2,4),(4,5)\}$, so $\mathcal{G}_1(v,w)$ is a directed path of length 3.

Definition 3.2. For a directed graph $\mathcal{G} = (V,E)$, the signed incidence matrix is the matrix $A_{\mathcal{G}}$ with rows indexed by V and columns indexed by E such that

$$(A_{\mathcal{G}})_{v_1,(v_2,v_3)} = \begin{cases} 1 & \text{if } v_1 = v_2 \\ -1 & \text{if } v_1 = v_3 \\ 0 & \text{otherwise.} \end{cases}$$

We recall the standard formula for the rank of the signed incidence matrix [14, Theorem 8.3.1].

Theorem 3.3. *Let $\mathcal{G} = (V,E)$ be a simple undirected graph with k connected components. Then the rank of $A_{\mathcal{G}}$ is $|V| - k$.*

For a directed graph $\mathcal{G} = (V,E)$, we denote by $\overline{\mathcal{G}}$ the undirected graph whose edge relation is the symmetric closure of E.

Corollary 3.4. *The rank of the signed incidence matrix of $\mathcal{G} = (V,E)$ is $|E|$ (i.e., the signed incidence matrix has full rank) if and only if every connected component of $\overline{\mathcal{G}}$ is a tree.*

Proof. For each connected component C, we have $|E(C)| = |V(C)| - 1$ if and only if C is a tree. Summing over the connected components yields the corollary. □

Corollary 3.5. *If $\overline{\mathcal{G}_1(v,w)}$ contains a cycle, then $M^{v,w}(x)$ does not have full rank for any $x \in N^{\vee}_{X(w),e_v} C(v)$.*

Proof. For $(a,b) \in E_1(v,w)$, observe that

$$M^{v,w}(x)_{(a,b),(r,s)} = \begin{cases} x_{v(a)v(b)} & \text{if } (r,s) = (a,a) \\ -x_{v(a)v(b)} & \text{if } (r,s) = (b,b) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the submatrix D of $M^{v,w}(x)$ with rows indexed by $E_1(v,w)$ and columns indexed by $E_0(v,w)$ is (up to scaling of rows) the transpose of the signed incidence matrix of $\mathcal{G}_1(v,w)$. In particular, the rank of D is at most the rank of the signed incidence matrix. In fact, because every nonzero entry of $M^{v,w}(x)$ in the rows $E_1(v,w)$ belongs to the submatrix D, then $M^{v,w}(x)$ does not have full rank if D does not have full rank. Now, if there is a cycle in $\overline{\mathcal{G}_1(v,w)}$, the signed incidence matrix of $\mathcal{G}_1(v,w)$ does not have full rank by Corollary 3.4. Thus D does not have full rank, so the result follows. □

We will use the preceding corollary to show that if $v \preceq w$ such that $w_0 w \in \text{maxsing}(w_0 v)$ in the 4231 or 3412 Cases, then $M^{v,w}(x)$ does not have full rank for any $x \in N^{\vee}_{X(w),e_v} C(v)$. We will do this by showing that in either case, the graph $\overline{\mathcal{G}_1(v,w)}$ contains a cycle.

Lemma 3.6. *Let $v \preceq w$ such that $w_0 w \in \text{maxsing}(w_0 v)$ in the 4231 Case. Then $M^{v,w}(x)$ does not have full rank for any x in $N^{\vee}_{X(w),e_v} C(v)$.*

Proof. Observe that $\text{Br}(w_0 w, w_0 v)$ has the form of Figure 2a. Thus, the Bruhat picture $\text{Br}(v,w)$ is as in Figure 4, via Remark 2.6.

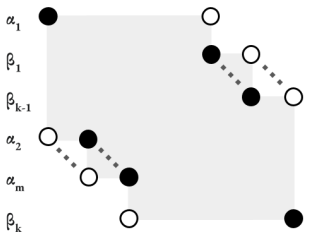


FIGURE 4. The Bruhat picture $\text{Br}(v,w)$ with $w_0 w \in \text{maxsing}(w_0 v)$ in the 4231 Case. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. The shaded region contains no filled or unfilled points.

Recalling that the shaded region in Figure 2a has no points in it, the shaded region of Figure 4 has no points in it. Applying Lemma 2.9, we can see

$$\begin{aligned} R(v,w) = &\{(\beta_i, \beta_{i+1}) : 1 \le i \le k-1\} \\ &\cup \{(\alpha_i, \alpha_{i+1}) : 1 \le i \le m-1\} \cup \{(\alpha_1, \beta_1), (\alpha_m, \beta_k)\}. \end{aligned}$$

Because the shaded area contains no dots, each of the above is minimal with respect to $\trianglelefteq$, and thus in $E_1(v,w)$. Therefore, the graph $\overline{\mathcal{G}_1(v,w)}$ is itself a cycle, and so $M^{v,w}(x)$ does not have full rank by Corollary 3.5. □

Lemma 3.7. *Let $v \preceq w$ such that $w_0 w \in \text{maxsing}(w_0 v)$ in the 3412 Case. Then $M^{v,w}(x)$ does not have full rank for any $x \in N^{\vee}_{X(w),e_v} C(v)$.*

Proof. If $w_0 w \in \text{maxsing}(w_0 v)$ in the 3412 case, the Bruhat picture $\text{Br}(v,w)$ is as in Figure 5, by applying Remark 2.6 to Figure 2b. This lets us determine

$$\begin{aligned} R(v,w) = &\{(\beta_i, \beta_{i+1}) : 1 \le i \le k-1\} \\ &\cup \{(\alpha_i, \alpha_{i+1}) : 1 \le i \le m-1\} \cup \{(\alpha_1, \beta_k), (\beta_1, \alpha_m)\}, \end{aligned}$$

and so by the same argument we made in Lemma 3.6, there is a cycle in $\overline{\mathcal{G}_1(v,w)}$. □

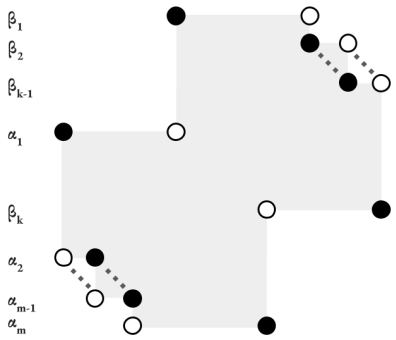


FIGURE 5. The Bruhat picture $\mathrm{Br}(v,w)$ with $w_0w \in \mathrm{maxsing}(w_0v)$ in the 3412 Case. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. The shaded region contains no filled or unfilled points.

3.2. **Maximal Singular Points in the 45312 Case.** In this subsection, we address the maximal singular points in the 45312 Case.

We first recall some results from [3].

Definition 3.8. For $w \in S_n$ and $Z = \{z_1, \ldots, z_k\} \subseteq \{1, \ldots, n\}$ such that $z_1 < \cdots < z_k$, we denote by $\mathrm{fl}_Z(w) \in S_{|Z|}$ the flattened permutation such that $\mathrm{fl}_Z(w)(i) < \mathrm{fl}_Z(w)(j)$ if and only if $w(z_i) < w(z_j)$. Moreover, for each pair $v \preceq w$, we define $\overline{v}^w = \mathrm{fl}_{E_0(v,w)}(v)$ and $\overline{w}^v = \mathrm{fl}_{E_0(v,w)}(w)$.

For instance, if $w = [2,4,5,3,1]$ and $Z = \{2,3,4\}$, then $\mathrm{fl}_Z(w) = [2,3,1]$.

The following lemma says that all relevant properties of the pair (v,w) are shared by the flattened pair $(\overline{v}^w, \overline{w}^v)$.

Lemma 3.9 ([3, Lemma 15, Proposition 16])**.** *We have $\overline{v} \preceq \overline{w}$ if and only if $v \preceq w$. Also, there exists a bijection between $R(v,w)$ and $R(\overline{v}^w, \overline{w}^v)$ which preserves $\trianglelefteq$. In particular, up to indexing of rows and columns, $M^{v,w}(x)$ is the same as $M^{\overline{v}^w, \overline{w}^v}(x)$.*

Lemma 3.10. *Let $v \preceq w$ such that $w_0w \in \mathrm{maxsing}(w_0v)$ in the 45312 Case. Then $M^{v,w}(x)$ does not have full rank for any $x \in N^\vee_{X(w),e_v}C(v)$.*

Proof. Using the Bruhat picture from Figure 3 and Remark 2.6, we see that $\mathrm{Br}(v,w)$ is given by Figure 6.

Recalling that the shaded region of Figure 3 contains no points besides the concentric circles depicted, the shaded region of Figure 6 also contains no points, with the exception of possible concentric circles in $\mathcal{A}_{\alpha_1,\beta_2}(w)$. Let M be the set of $\mu \in (\alpha_1, \beta_2)$ such that $\mathrm{pt}_v(\mu) = \mathrm{pt}_w(\mu)$ is one of said concentric circles, i.e.

$$M = \{\mu \in (\alpha_1, \beta_2) : v(\alpha_1) < v(\mu) = w(\mu) < v(\beta_2)\}.$$

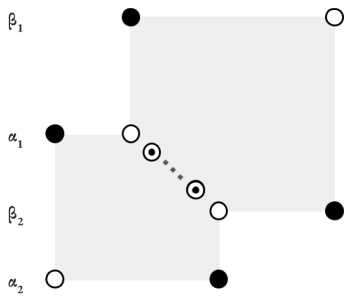


FIGURE 6. The Bruhat picture $\mathrm{Br}(v,w)$ with $w_0w \in \mathrm{maxsing}(w_0v)$ in the 45312 Case. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. The shaded region contains no filled or unfilled points other than the indicated concentric circles.

Enumerate $M = \{\mu_1, \ldots, \mu_n\}$ such that $\mu_1 < \cdots < \mu_n$. We have

$$\begin{aligned} R(v,w) = &\{(\beta_1, \mu_i) : \mu_i \in M\} \cup \{(\alpha_1, \mu_i) : \mu_i \in M\} \\ &\cup \{(\mu_i, \beta_2) : \mu_i \in M\} \cup \{(\mu_i, \alpha_2) : \mu_i \in M\} \\ &\cup \{(\mu_i, \mu_j) : i < j\} \cup \{(\beta_1, \beta_2), (\beta_1, \alpha_2), (\alpha_1, \beta_2), (\alpha_1, \alpha_2)\}, \end{aligned}$$

so

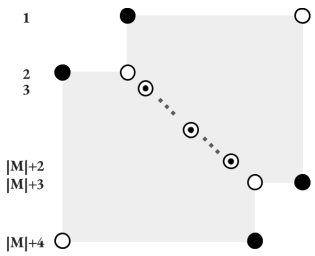
$$E_0(v,w) = \{\beta_1, \alpha_1, \beta_2, \alpha_2\} \cup M.$$

Thus $\mathrm{Br}(\overline{v}, \overline{w})$ is depicted in Figure 7a by Lemma 3.9. Furthermore, the Bruhat picture $\mathrm{Br}(\overline{v}^w, w_0)$ of flattened permutations in $S_{|M|+4}$ is as in Figure 7b (strictly speaking, what is depicted is the case where $|M|$ is odd, but the difference is immaterial).

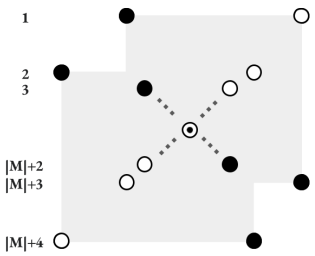
We observe that Figure 7a and Figure 7b have the same shaded regions. Thus, $R(\overline{v}^w, \overline{w}^v) = R(\overline{v}^w, w_0)$, so $M^{\overline{v}^w, \overline{w}^v}(x)$ and $M^{\overline{v}^w, w_0}(x)$ are in fact the same matrix. Recall [25, Theorem 1.2], which says that if $v \preceq w$ and w is 231-avoiding, then $X(w_0v)$ is smooth at e_{w_0w} if and only if there is an open B-orbit in $N^\vee_{X(w)}C(v)$. Strictly speaking, it is stated in the setting of G-orbits in $G/B \times G/B$, but the setting of B-orbits in G/B is equivalent. The permutation w_0 is 231-avoiding, and $X(w_0\overline{v}^w)$ is singular at $e_{w_0\overline{w}^v}$ by Remark 2.6 and Theorem 2.11. Thus, the matrices $M^{\overline{v}^w, w_0}(x) = M^{\overline{v}^w, \overline{w}^v}(x)$ are the same for all $x \in N^\vee_{X(w), e_v}C(v)$ and are not of full rank. Finally, by Lemma 3.9, $M^{v,w}(x)$ is not of full rank for all $x \in N^\vee_{X(w),e_v}C(v)$. □

Corollary 3.11. *If $v \preceq w$ and $w_0w \in \mathrm{maxsing}(w_0v)$, then the action of B_v on the conormal space $N^\vee_{X(w),e_v}C(v)$ does not admit an open orbit.*

Proof. By Theorem 2.11, we have $w_0w \in \mathrm{maxsing}(w_0v)$ in the 4231 Case, the 3412 Case, or the 45312 Case. These are addressed in turn in Lemma 3.6, Lemma 3.7, and Lemma 3.10. □



(A) The Bruhat picture $\mathrm{Br}(\overline{v}^w, \overline{w}^v)$ with $w_0 w \in \mathrm{maxsing}(w_0 v)$ in the 45312 Case. On the shaded region, $d_{\overline{v}^w, \overline{w}^v} \equiv 1$, and outside of the shaded region, $d_{\overline{v}^w, \overline{w}^v} \equiv 0$.



(B) The Bruhat picture $\mathrm{Br}(\overline{v}^w, w_0)$ with $w_0 w \in \mathrm{maxsing}(w_0 v)$ in the 45312 Case. On the shaded region, $d_{\overline{v}^w, w_0} \geq 1$, and outside of the shaded region, $d_{\overline{v}^w, w_0} \equiv 0$. By the dotted lines going from the top left to bottom right (resp. top right to bottom left), we mean that the filled dots continue down and to the right (resp. down and to the left) in order.

FIGURE 7

Theorem 3.12. *Let $v \preceq w$ such that $X(w_0 v)$ is singular at $e_{w_0 w}$. Then there is no open B_v-orbit in $N^\vee_{X(w), e_v} C(v)$.*

Proof. Let $v \preceq w' \preceq w$. Then the natural projection $N^\vee_{X(w), e_v} C(v) \twoheadrightarrow N^\vee_{X(w'), e_v} C(v)$ is surjective, linear, and B_v-equivariant. Thus, because both the domain and codomain of this map are finite-dimensional, the map is a projection up to a change of coordinates, and is thus an open map. Therefore, if there is an open B_v-orbit in $N^\vee_{X(w), e_v} C(v)$, then there is also an open orbit in $N^\vee_{X(w'), e_v} C(v)$. Equivalently, if there is no open B_v-orbit in $N^\vee_{X(w'), e_v} C(v)$, there is no open B_v-orbit in $N^\vee_{X(w), e_v} C(v)$. For every $v \preceq w$ such that $X(w_0 v)$ is singular at $e_{w_0 w}$, there exists some $w' \preceq w$ such that $w_0 w' \in \mathrm{maxsing}(w_0 v)$. By Corollary 3.11, there is no open B_v-orbit in $N^\vee_{X(w'), e_v} C(v)$, so there is no open B_v-orbit in $N^\vee_{X(w), e_v} C(v)$. □

4. A Necessary Condition for Non-Existence of an Open B_v-orbit

4.1. **Defining the Graphs.** In this subsection, we recursively define graphs analogous to $\mathcal{G}_1(v, w)$ which capture the remaining information of $R(v, w)$. The idea is to produce a filtration on $R(v, w)$ based on how many elements are less than a fixed element with respect to $\trianglelefteq$. Recall our construction of $E_0(v, w)$ and $E_1(v, w)$ in Section 3.1.

Definition 4.1. Given $E_i(v, w) \subseteq R(v, w)$, we define $E_{i+1}(v, w)$ as the elements of

$$R(v, w) \setminus \left(\bigcup_{1 \leq j \leq i} E_j(v, w) \right)$$

which are minimal with respect to $\trianglelefteq$.

For instance, in Figure 1b, we have

$$E_0(v,w) = \{(1,1),(2,2),(4,4),(5,5)\},$$
$$E_1(v,w) = \{(1,2),(2,4),(4,5)\},$$

and

$$E_2(v,w) = \{(2,5)\}.$$

These are visualized in Figure 8, in which the red arrows denote elements of $E_1(v,w)$ and the blue arrow is the element of $E_2(v,w)$.

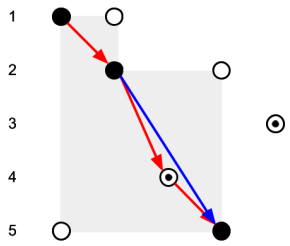


FIGURE 8. The Bruhat picture $\mathrm{Br}(v,w)$ with $v = [1,2,5,3,4]$ and $w = [2,4,5,3,1]$. On the shaded region, $d_{v,w} \equiv 1$, and outside of the shaded region, $d_{v,w} \equiv 0$. For each (i,j) in $E_1(v,w)$, there is a red arrow from $\mathrm{pt}_i(v)$ to $\mathrm{pt}_j(v)$, and for each (i,j) in $E_2(v,w)$, there is a blue arrow from $\mathrm{pt}_i(v)$ to $\mathrm{pt}_j(v)$.

Whenever $j > i$, we have $M^{v,w}(x)_{(a,b),(r,s)} = 0$ for any $(a,b) \in E_i(v,w)$ and $(r,s) \in E_j(v,w)$. Thus, after removing all columns which are identically zero and permuting rows and columns, we can write $M^{v,w}(x)$ as a block matrix

$$M^{v,w}(x) = \begin{bmatrix} D_1 & & & & \\ * & D_2 & & & \\ * & * & D_3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & * & * & \cdots & D_m \end{bmatrix}$$

where the rows of D_i are indexed by $E_i(v,w)$ and the columns of D_i are indexed by $E_{i-1}(v,w)$. Here, m is the greatest value such that $E_m(v,w)$ is nonempty.

Thus, we have

$$\operatorname{rank}(M^{v,w}(x)) \geq \sum_{1\leq i\leq m} \operatorname{rank}(D_i).$$

Lemma 4.2. *If $\overline{\mathcal{G}_1(v,w)}$ is a forest, then for every $(a,b) \in E_i(v,w)$, there are exactly $i-1$ values k such that $\mathrm{pt}_v(k) \in \mathcal{A}_{ab}(v)$. Further, for every $m_1 < m_2$ such that $\mathrm{pt}_v(k_1), \mathrm{pt}_v(k_2) \in \mathcal{A}_{ab}(v)$, we have $v(k_1) < v(k_2)$.*

Proof. We induct on i. The $i = 1$ is clear. Suppose $i \geq 2$, and assume the lemma for every $j < i$. Let p be the minimum k in (a,b) such that $\mathrm{pt}_v(k) \in \mathcal{A}_{ab}(v)$.

If there exists q such that $p < q < b$ and $v(a) < v(q) < v(p)$, then take a minimal such q. We have, by minimality, that (a,p) and (a,q) are in $E_1(v,w)$. Further, by the induction hypothesis, (p,b) and (q,b) are in $E_{j_p}(v,w)$ and $E_{j_q}(v,w)$ respectively, with $j_p, j_q < i$. Applying the induction hypothesis to (p,b) and (q,b), we have $j_p - 1$ and $j_q - 1$ elements $k_1^p, \ldots, k_{j_p-1}^p$ and $k_1^q, \ldots, k_{j_q-1}^q$ of $\mathcal{A}_{pb}(v)$ and $\mathcal{A}_{qb}(v)$ respectively. Letting $k_0^p = p$, $k_0^q = q$, and $k_{j_p}^p = k_{j_q}^q = b$, we must have $(k_r^p, k_{r+1}^p) \in E_1(v,w)$ and $(k_r^q, k_r^{q+1}) \in E_1(v,w)$ for every r, so these give directed paths in $\mathcal{G}_1(v,w)$ from p to b and from q to b. Together with the edges $(a,p) \in E_1(v,w)$ and $(a,q) \in E_1(v,w)$, there is a cycle in $\overline{\mathcal{G}_1(v,w)}$.

Thus, there is no such q, so $(p,b) \in E_{i-1}(v,w)$, because (a,b) is in $E_i(v,w)$ and is a minimal element with respect to $\trianglelefteq$ of

$$\{(i,j) \in R(v,w) : (p,b) \trianglelefteq (i,j)\}.$$

The $k_1^p, \ldots, k_{j_p-1}^p$ form the remaining $i-2$ points in $\mathcal{A}_{ab}(v)$. □

Definition 4.3. If $\overline{\mathcal{G}_1(v,w)}$ is a forest, then for $i \geq 2$, we can define a directed graph $\mathcal{G}_i(v,w)$ with vertices labeled by $E_{i-1}(v,w)$ and edges labeled by $E_i(v,w)$, such that the vertex $(r,s) \in E_{i-1}(v,w)$ is the source (resp. target) of the edge (a,b) if and only if $(r,s) \trianglelefteq (a,b)$ and $r = a$ (resp. $s = b$). Observe that D_i is the signed incidence matrix of $\mathcal{G}_i(v,w)$ when $x = \vec{1}$.

Proposition 4.4. *Let $v \preceq w$. If every connected component of every $\overline{\mathcal{G}_i(v,w)}$ is a tree, then $M^{v,w}(\vec{1})$ has full rank.*

Proof. The submatrix $D = D_1$ is the signed incidence matrix of $\mathcal{G}_1(v,w)$, and in fact, D_i is the signed incidence matrix of $\mathcal{G}_i(v,w)$. Thus, by Corollary 3.4, each D_i has full rank, and so $M^{v,w}(\vec{1})$ does as well. □

4.2. **The Necessary Condition.** In this subsection, we show that the graphs $\mathcal{G}_j(v,w)$ allow us to characterize the configurations in $\mathrm{Br}(v,w)$ whenever there is no open B_v-orbit in $N^\vee_{X(w),e_v} C(v)$.

Definition 4.5. Let $\mathcal{G} = (V,E)$ be a directed graph. The line digraph $L(\mathcal{G})$ has vertex set E, and its edges are of the form $((v_1,v_2),(v_2,v_3))$ with $(v_1,v_2),(v_2,v_3) \in E$.

The definition for undirected graphs is analogous.

We now recall a theorem of Chartrand on line graphs of trees [15, Theorem 8.5].

Theorem 4.6 (Chartrand)**.** *The line graph $L(\mathcal{T})$ of an undirected tree $\mathcal{T}$ is a block graph. That is, every bi-connected subset (a subset $A \subseteq T$ such that for every $v \in A$, the induced subgraph $A \setminus \{v\} \subseteq T$ is connected) is a clique.*

The following graph-theoretic lemma will allow us to show that, if $\overline{\mathcal{G}_1(v,w)}$ is a forest, then a cycle in $\overline{\mathcal{G}_i(v,w)}$ with $i > 1$ induces a configuration in $\mathrm{Br}(v,w)$ akin to Figure 3.

Lemma 4.7. *Let $\mathcal{G} = (V, E)$ be a directed graph such $\overline{\mathcal{G}}$ is a forest, and let $\mathcal{H}$ be a directed subgraph of $L(\mathcal{G})$ such that $\overline{\mathcal{H}}$ contains a cycle. Then there is some c in $\mathcal{G}$ such that every element of the cycle is an edge (v, c) or an edge (c, v) for some v in $\mathcal{H}$. Further, every edge in the cycle is from a vertex (v, c) to a vertex (c, v').*

Proof. Take some cycle in $\overline{\mathcal{H}}$ with vertices $(v_1, v_2), \ldots, (v_{n-1}, v_n)$. Then the set

$$\{(v_1, v_2), \ldots, (v_{n-1}, v_n)\}$$

is bi-connected, so by Theorem 4.6, the induced subgraph $\{(v_1, v_2), \ldots, (v_{n-1}, v_n)\} \subseteq L(\overline{\mathcal{G}})$ is a clique. Therefore, there exists an element c of $\{v_1, v_2\}$ such that $v_i = c$ or $v_{i+1} = c$ for every i. Without loss of generality, $c = v_1$. Let (v_i', v_{i+1}') be the directed edge in $\mathcal{G}$ corresponding to the node (v_i, v_{i+1}) in $\overline{L(\mathcal{G})}$. Taking some edge in the cycle from (v_i', v_{i+1}') to (v_{i+2}', v_{i+3}'), we have $v_{i+1}' = v_{i+2}'$. Further, we must have $c \in \{v_i', v_{i+1}'\}$ and $c \in \{v_{i+2}', v_{i+3}'\}$, so $c = v_{i+1}' = v_{i+2}'$. □

The following is the main result of the section, and describes the configurations that must occur in the $\mathrm{Br}(v,w)$ when there is no open B_v-orbit in $N^\vee_{X(w),e_v} C(v)$.

Theorem 4.8. *Let $v \preceq w$. If $M^{v,w}(x)$ does not have full rank for every $x \in N^\vee_{X(w),e_v} C(v)$, then $\overline{\mathcal{G}_1(v,w)}$ has a cycle or there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that the configuration of Figure 9 occurs in $\mathrm{Br}(v,w)$.*

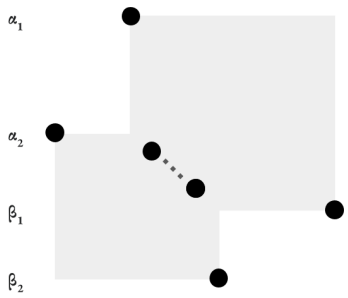


FIGURE 9. On the shaded region, $d_{v,w}$ is positive, and the shaded region contains no filled dots other than the indicated ones. By the dotted line, we mean that the filled dots continue down and to the right in order.

Proof. If every connected component of every $\overline{\mathcal{G}_i(v,w)}$ is a tree, we know that $M^{v,w}(\vec{1})$ has full rank by Proposition 4.4. Thus it remains to show that $\alpha_1, \alpha_2, \beta_1, \beta_2$ in the configuration of Figure 9 exist when $\overline{\mathcal{G}_1(v,w)}$ is a forest and some $\overline{\mathcal{G}_i(v,w)}$ contains a cycle. Take $j > 1$ to be the smallest index such that $\overline{\mathcal{G}_j(v,w)}$ contains a cycle. The digraph $\mathcal{G}_j(v,w)$ is a subgraph of the line digraph of $\mathcal{G}_{j-1}(v,w)$, so by Lemma 4.7, there is some $(r,s) \in E_{j-2}(v,w)$ such that every vertex of the cycle is labeled by a vertex of the form $(a,s) \in E_{j-1}(v,w)$ or $(r,b) \in E_{j-1}(v,w)$. Further, every edge in the cycle is from some (a,s) to some (r,b). Choose the minimum such a, and call it α_1. The two edges in the cycle containing (α_1, s) must be of the form (α_1, β_1) and (α_1, β_2). Without loss of generality, $\beta_1 < \beta_2$. The other edge of the cycle containing β_2 must be of the form (α_2, β_2) for some α_2. We have $\alpha_1 < \alpha_2$ by the construction of α_1, and we have $\alpha_2 < r \leq s < \beta_1$. Also, both $\mathcal{A}_{\alpha_1\beta_1}(v)$ and $\mathcal{A}_{\alpha_2\beta_2}(v)$ are shaded in $\mathrm{Br}(v,w)$.

Because $\overline{\mathcal{G}_1(v,w)}$ is a forest, Lemma 4.2 implies that $\mathrm{Br}(v,w)$ contains the configuration in Figure 9, with the central filled dots forming $(r,s) \in E_{j-2}(v,w)$. To see why, consider the following arguments.

(1) The inequalities we have already established give us the correct relative positions of α_1, α_2, β_1, β_2, r, and s, as indicated in Figure 9.
(2) The shading is accurate, because the entire region is contained in $\mathcal{A}_{\alpha_1\beta_1}(v)$ and $\mathcal{A}_{\alpha_2\beta_2}(v)$, as can be seen in Figure 9.
(3) The filled dots continuing down and to the right in order is a consequence of Lemma 4.2 and (r,s) being in $E_{j-2}(v,w)$. Similarly, there are exactly $j-1$ filled dots in $(r,s) \times (v(r), v(s))$, and the dots are in the depicted order.
(4) There are no filled dots in the shaded region other than the ones indicated in Figure 9. This is because, in each of $\mathcal{A}_{\alpha_1\beta_1}(v)$ and $\mathcal{A}_{\alpha_2\beta_2}(v)$, there are already $(j-2)+2 = j$ indicated dots; thus, having any more would contradict $(\alpha_1, \beta_1) \in E_j(v,w)$ or $(\alpha_2, \beta_2) \in E_j(v,w)$. □

5. Proof of the Main Theorem

In this section, we prove Theorem 1.2. In light of Theorem 3.12 and Theorem 4.8, it remains to show that, if $X(w)$ is factorial and there is cycle in some $\overline{\mathcal{G}_i(v,w)}$, then $X(w_0 v)$ is singular at $e_{w_0 w}$. In Section 5.1, we prove some preliminary lemmas, which in combination with Theorem 4.8 relate singularity to open orbits. In Section 5.2, we complete the proof of this final implication, using a result of Dyer on the coefficient of q in the Kazhdan–Lusztig polynomials $P_{v,w}(q)$.

5.1. Preliminary Lemmas.

Here, we will prove several auxiliary lemmas about Bruhat pictures which will be useful in our proof of Theorem 1.2.

Lemma 5.1. *Let $v \preceq w$ and $v' \preceq w$. Then $v \preceq v'$ if and only if $d_{v',w}(p,q) \leq d_{v,w}(p,q)$ for all $p, q \in \mathbb{Z}$.*

Proof. This is simply because, for all p, q,

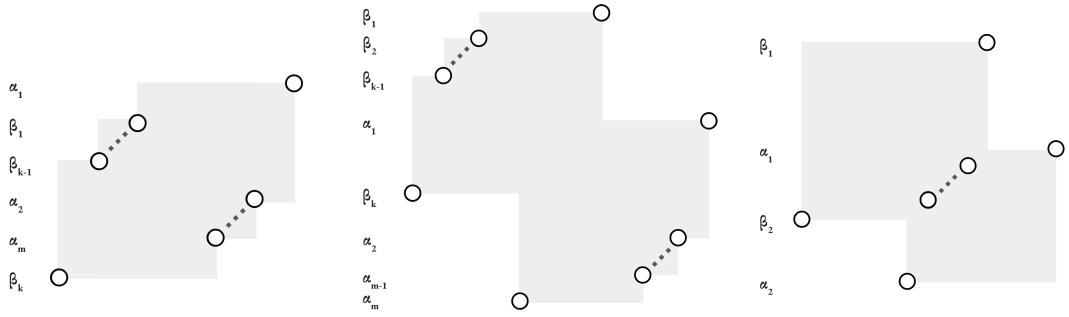
$$r_w(p,q) - r_{v'}(p,q) \leq r_w(p,q) - r_v(p,q)$$

is equivalent to $d_{v,v'}(p,q) \geq 0$, which is equivalent to $v \preceq v'$. □

The following corollary gives an interpretation of the singularity of $X(w)$ at e_v in terms of $\mathrm{Br}(v, w)$. This ties the occurrence of Figure 9 in Theorem 4.8 to singularity.

Corollary 5.2. *Let $v \preceq w$. Then $X(w)$ is singular at e_v if and only if there exist integers $\alpha_1, \dots \alpha_m$ and $\beta_1, \dots, \beta_k$ in $[n]$ such that the unfilled dots in these rows are in the configurations of Figure 10a, Figure 10b, or Figure 10c. Further,*

1. *every indicated region is shaded;*
2. *the shaded regions contain no unfilled dots, other than possibly in the rectangle $(\alpha_1, \beta_2) \times (v(\alpha_2), v(\beta_1))$, as depicted in Figure 10c; and*
3. *in Figure 10c, the unfilled dots in $(\alpha_1, \beta_2) \times (v(\alpha_2), v(\beta_1))$ must proceed down and to the left in order.*



(A) (B) (C)

FIGURE 10

Proof. Note that $X(w)$ is singular at e_v if and only if there exists $v' \succeq v$ such that $v' \in \mathrm{maxsing}(w)$.

First, assume that such a v' exists. By Theorem 2.11, we have $v' = w \circ c$, with $c = (\alpha_1, \dots, \alpha_m, \beta_k, \dots, \beta_1)$, such that the pair (v', w) satisfies the conditions of some case of Theorem 2.11. Then the unfilled dots in $\mathrm{Br}(v, w)$ will be in the same places as the unfilled dots in $\mathrm{Br}(v', w)$ by Remark 2.6. Also, because $v \preceq v'$ and the indicated area is shaded in $\mathrm{Br}(v', w)$, then the indicated area is shaded in $\mathrm{Br}(v, w)$ by Lemma 5.1.

Conversely, assume that one of Figure 10a, Figure 10b, or Figure 10c occurs in $\mathrm{Br}(v, w)$. Then letting $c = (\alpha_1, \dots, \alpha_m, \beta_k, \dots, \beta_1)$ and $v' = w \circ c$, we have $d_{v',w} \equiv 1$ on the shaded region of the relevant figure, and $d_{v',w} \equiv 0$ outside of that shaded region. By Lemma 2.9, we have $d_{v,w} \geq 0$ outside of the shaded region, and by the definition of shading we have $d_{v,w} \geq 1$ on the shaded region. This gives $d_{v,w} \geq d_{v',w}$, and applying Lemma 5.1 yields $v \preceq v'$. Finally, $v' \in \mathrm{maxsing}(w)$ by Theorem 2.11, so $X(w)$ is singular at e_v. □

Lemma 5.3. *Let $v \preceq w$. If $\mathcal{A}_{ij}(v)$ is shaded in $\mathrm{Br}(v, w)$, then $(i, j) \times (w_0 v(j), w_0 v(i))$ is shaded in $\mathrm{Br}(w_0 w, w_0 v)$.*

Proof. By definition, for any $p, q \in \mathbb{Z}$, we have

$$\begin{aligned} r_{w_0 v}(p, w_0(q)+1) &= \#\{i \le p : w_0 v(i) \ge w_0(q)+1\} \\ &= \#\{i \le p : n - v(i) + 1 \ge n - q + 2\} \\ &= \#\{i \le p : v(i) \le q-1\} \\ &= p - r_v(p,q), \end{aligned}$$

so

$$\begin{aligned} d_{w_0 w, w_0 v}(p, w_0(q)+1) &= r_{w_0 v}(p, w_0(q)+1) - r_{w_0 w}(p, w_0(q)+1) \\ &= (p - r_v(p,q)) - (p - r_w(p,q)) \\ &= d_{v,w}(p,q). \end{aligned}$$

Now let $p \in (i,j)$ and $q \in (v(i), v(j)]$. Again by definition, we have $r_v(p,q) = r_{vs_{ij}}(p,q) - 1$. Because $\mathcal{A}_{ij}(v)$ is shaded in $\mathrm{Br}(v,w)$, we have $v \prec v s_{ij} \preceq w$, so

$$\begin{aligned} d_{v,w}(p,q) &= r_w(p,q) - r_v(p,q) \\ &= r_w(p,q) - r_{vs_{ij}}(p,q) + 1 \\ &= d_{vs_{ij},w}(p,q) + 1. \end{aligned}$$

By Lemma 2.9, we have $d_{vs_{ij},w}(p',q') \ge 0$ for all $p', q' \in \mathbb{Z}$, so $d_{v,w}(p,q) \ge 1$.

Finally, take (p'', q'') in $(i,j) \times (w_0 v(j), w_0 v(i))$. Then $(p'', w_0(q'')+1)$ is in $(i,j) \times (v(i), v(j)]$, so

$$d_{w_0 w, w_0 v}(p'', q'') = d_{v,w}(p'', w_0(q'')+1) \ge 1. \qquad \square$$

Corollary 5.2 and Lemma 5.3 give a characterization of the existence of an open B_v-orbit in $N^\vee_{X(w),e_v} C(v)$ in terms of the graphs $\mathcal{G}_i(v,w)$, which is of independent interest.

Theorem 5.4. *Let $v \preceq w$. Then $M^{v,w}(x)$ has full rank for some $x \in N^\vee_{X(w),e_v} C(v)$ if and only if every $\overline{\mathcal{G}_i(v,w)}$ is a forest. In fact, this equivalence gives a stronger version of Theorem 2.4, in the sense that the existence of an open B_v-orbit in $N^\vee_{X(w),e_v} C(v)$ is equivalent to $M^{v,w}(\vec{1})$ having full rank.*

Proof. By Corollary 3.5, if $\overline{\mathcal{G}_1(v,w)}$ contains a cycle, then $M^{v,w}(x)$ does not have full rank for any $x \in N^\vee_{X(w),e_v} C(v)$. If $\overline{\mathcal{G}_1(v,w)}$ is a forest and $\overline{\mathcal{G}_i(v,w)}$ for $i > 1$ contains a cycle, then $\mathrm{Br}(v,w)$ contains the configuration in Figure 9 by the argument in the proof of Theorem 4.8. By Remark 2.6, Lemma 5.3, and Corollary 5.2, $X(w_0 v)$ is singular at $e_{w_0 w}$, so $M^{v,w}(x)$ does not have full rank for any $x \in N^\vee_{X(w),e_v} C(v)$ by Theorem 3.12. Conversely, if every $\overline{\mathcal{G}_i(v,w)}$ is a forest, then $M^{v,w}(\vec{1})$ has full rank by Proposition 4.4.

For the second equivalence, Proposition 4.4 implies that if $M^{v,w}(\vec{1})$ does not have full rank, then there is some cycle in some $\overline{\mathcal{G}_i(v,w)}$, and we are done. $\square$

Remark 5.5. Theorem 5.4 yields an $O(n^2)$ algorithm for determining whether there is an open B-orbit in $N^\vee_{X(w)} C(v)$. The steps are as follows.

(1) Compute $d_{v,w}(p,q)$ for every $(p,q) \in [n]^2$.
(2) Compute $R(v,w)$ using the computed $d_{v,w}$. We do this in $O(n^2)$ by first computing the function
$$f(p,q) = \#\{(p',q') \in [1,p] \times [1,q] : d_{v,w}(p',q') = 0\}$$
for each $(p,q) \in [n]^2$, and then checking whether
$$\begin{aligned}\#\{(p,q) \in \mathcal{A}_{ij}(v) : d_{v,w}(p,q) = 0\} =& f(i,v(i)) + f(j-1,v(j)-1) \\ &- f(i,v(j)-1) - f(j-1,v(i))\end{aligned}$$
is 0 for each pair $i < j$.
(3) Compute $E_1(v,w)$ and determine whether there is a cycle in $\overline{\mathcal{G}_1(v,w)}$. We do this in $O(n^2)$ because finding a cycle in a graph $\mathcal{G} = (V,E)$ can be done in $O(|V| + |E|)$.
(4) If $\overline{\mathcal{G}_1(v,w)}$ is a forest, sort the elements of $R(v,w)$ into each $E_i(v,w)$. We do this in $O(n^2)$ using Lemma 4.2.
(5) Check whether each $\overline{G_i(v,w)}$ contains a cycle. Again, this is $O(n^2)$ because finding a cycle in each takes $O(|E_i(v,w)| + |E_{i-1}(v,w)|)$.

5.2. **Proof of Theorem 1.2.** In this subsection, we state a series of corollaries and equivalent formulations of results of Deodhar, Dyer, Tsukerman–Williams, and Bousquet–Mélou–Butler. We then prove the main theorem Theorem 1.2.

Recall that there is a family of polynomials $P_{v,w}$ which was introduced by Kazhdan–Lusztig in [18, 19], indexed by the pairs $v \preceq w$ of elements of S_n. Further, recall that $P_{v,w} = 1$ if and only if $X(w)$ is smooth at e_v by a result of Deodhar [8].

We establish the following straightforward consequence of a result of Dyer in [9].

Theorem 5.6. *For $v \preceq w$, we have*
$$[q]P_{v,w} \geq |E_1(w_0 w, w_0 v)| - |E_1(v,w)|.$$

Proof. In [9], $|E_1(w_0 w, w_0 v)|$ is denoted by $c_{v,w}$, and $|E_1(v,w)|$ is denoted by $a_{v,w}$. The main theorem of [9] says that there is an integer $\Delta_{v,w}$ such that $|E_1(v,w)| - \Delta_{v,w} \geq 0$, and that
$$[q]P_{v,w} = |E_1(w_0 w, w_0 v)| - \Delta_{v,w}.$$
Together, these yield the theorem. □

The following is a result of Bousquet–Mélou–Butler, Theorem 1 of [5]. More specifically, as is noted in the introduction of [5], the two theorems are equivalent by Proposition 2 of [28].

Theorem 5.7. *Let $w \in S_n$. Then the following are equivalent.*
(1) *$X(w)$ is factorial.*
(2) *The graph $\overline{\mathcal{G}_1(w_0 w, w_0)}$ is a forest.*
(3) *w avoids 4231 and $3\underline{41}2$, where the underline means that the 4 and 1 must be adjacent in the one line notation for w.*

Remark 5.8. Because smoothness of $X(w)$ is equivalent to w avoiding 4231 and 3412, the first example of a permutation w such that $X(w)$ is factorial but not smooth is 45312 in S_5.

The atom and coatom graphs, defined by Tsukerman–Williams in [27], are obtained from $\mathcal{G}_1(v,w)$ and $\mathcal{G}_1(w_0w,w_0v)$ by adding isolated vertices. Proposition 4.10 of [27] says that the atom and coatom graphs have the same partition into connected components, and thus the same set of isolated vertices. Disregarding these vertices gives the following result.

Theorem 5.9. *For $v \preceq w$, the graphs $\overline{\mathcal{G}_1(v,w)}$ and $\overline{\mathcal{G}_1(w_0w,w_0v)}$ have the same number of connected components.*

We now prove our main theorem, which is a rephrasing of Theorem 1.2.

Theorem 5.10. *Let $v \preceq w$, and let $X(w)$ be factorial. Then there is an open B_v-orbit in $N^\vee_{X(w),e_v}C(v)$ if and only if $X(w_0v)$ is smooth at e_{w_0w}.*

Proof. One implication is handled in Theorem 3.12. It is then left only to show that if there is no open B_v-orbit in $N^\vee_{X(w),e_v}C(v)$, then $X(w_0v)$ is singular at e_{w_0w}. By Theorem 4.8, if there is no open B_v-orbit in $N^\vee_{X(w),e_v}C(v)$, then $\overline{\mathcal{G}_1(v,w)}$ contains a cycle or $\overline{\mathcal{G}_1(v,w)}$ is a forest and $\mathrm{Br}(v,w)$ contains the configuration in Figure 9.

(1) We begin with the case in which $\overline{\mathcal{G}_1(v,w)}$ is a forest and we have an occurrence of the configuration in Figure 9 in $\mathrm{Br}(v,w)$. By Remark 2.6 and Lemma 5.3, applying w_0 to the pair (v,w) preserves the shading, the lack of dots in the shaded region, and the relative positions of the dots, so we have the configuration in Figure 10c in $\mathrm{Br}(w_0w,w_0v)$. Thus, by Corollary 5.2, $X(w_0v)$ is singular at e_{w_0w}.

(2) We now handle the case in which there is a cycle in $\overline{\mathcal{G}_1(v,w)}$. Because $X(w)$ is factorial, the graph $\overline{\mathcal{G}_1(w_0w,w_0)}$ is a forest by Theorem 5.7. The graph $\overline{\mathcal{G}_1(w_0w,w_0v)}$ is a subgraph of $\overline{\mathcal{G}_1(w_0w,w_0)}$, so it is a forest as well. Therefore,

$$|E_1(w_0w,w_0v)| = \mathrm{rank}(A_{\mathcal{G}_1(w_0w,w_0v)})$$

by Corollary 3.4. Similarly, because $\overline{\mathcal{G}_1(v,w)}$ contains a cycle, we have

$$|E_1(v,w)| > \mathrm{rank}(A_{\mathcal{G}_1(v,w)})$$

by Corollary 3.5. Combining Theorem 5.9 and Theorem 3.3, we have

$$\mathrm{rank}(A_{\mathcal{G}_1(v,w)}) = \mathrm{rank}(A_{\mathcal{G}_1(w_0w,w_0v)}),$$

so

$$\begin{aligned}|E_1(w_0w,w_0v)| &= \mathrm{rank}(A_{\mathcal{G}_1(v,w)})\\ &= \mathrm{rank}(A_{\mathcal{G}_1(w_0w,w_0v)})\\ &< |E_1(v,w)|.\end{aligned}$$

Applying Theorem 5.6 to the pair (w_0w,w_0v), this gives $[q]P_{w_0w,w_0v} > 0$. For all $v \preceq w$, we have $P_{v,w} = 1$ if and only if $X(w)$ is smooth at e_v by the main theorem of [8], so $X(w_0v)$ is not smooth at e_{w_0w}. □

Remark 5.11. In fact, only the second case in the second implication in the proof of Theorem 5.10 requires the factoriality hypothesis.

References

[1] Avraham Aizenbud and Erez Lapid. A binary operation on irreducible components of Lusztig's nilpotent varieties I: Definition and properties. *Pure and Applied Mathematics Quarterly*, 21(1):5–41, 2025. doi:10.4310/PAMQ.241203024730.

[2] Alexander Beilinson and Joseph Bernstein. Localisation de $\mathfrak{g}$-modules. *Comptes rendus des séances de l'Académie des sciences. Série 1, Mathématique*, 292(1):15–18, 1981.

[3] Sara C. Billey and Gregory S. Warrington. Maximal singular loci of Schubert varieties in SL(n)/b. *Transactions of the American Mathematical Society*, 355(10):3915–3945, 2003. arXiv:math/0102168, doi:10.1090/S0002-9947-03-03019-8.

[4] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, Berlin, Heidelberg, 2005. doi:10.1007/3-540-27596-7.

[5] Mireille Bousquet-Mélou and Steve Butler. Forest-like permutations. *Annals of Combinatorics*, 11(3–4):335–354, 2007. doi:10.1007/s00026-007-0322-1.

[6] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan–Lusztig conjecture and holonomic systems. *Inventiones Mathematicae*, 64(3):387–410, 1981. doi:10.1007/BF01389272.

[7] Aurélie Cortez. Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire. *Advances in Mathematics*, 178(2):396–445, 2003. doi:10.1016/S0001-8708(02)00081-6.

[8] Vinay V. Deodhar. Local Poincaré duality and non-singularity of Schubert varieties. *Communications in Algebra*, 13(6):1379–1388, 1985. doi:10.1080/00927878508823227.

[9] Matthew J. Dyer. On coefficients of q in Kazhdan–Lusztig polynomials. In Gus I. Lehrer, editor, *Algebraic Groups and Lie Groups*, volume 9 of *Australian Mathematical Society Lecture Series*, pages 189–194. Cambridge University Press, Cambridge, 1997.

[10] Christof Geiß, Bernard Leclerc, and Jan Schröer. Semicanonical bases and preprojective algebras. *Annales Scientifiques de l'École Normale Supérieure*, 38(2):193–253, 2005. doi:10.1016/j.ansens.2004.12.001.

[11] Christof Geiß, Bernard Leclerc, and Jan Schröer. Rigid modules over preprojective algebras. *Inventiones Mathematicae*, 165(3):589–632, 2006. doi:10.1007/s00222-006-0507-y.

[12] Christof Geiß, Bernard Leclerc, and Jan Schröer. Kac–Moody groups and cluster algebras. *Advances in Mathematics*, 228(1):329–433, 2011. doi:10.1016/j.aim.2011.05.011.

[13] Christof Geiß and Jan Schröer. Extension-orthogonal components of preprojective varieties. *Transactions of the American Mathematical Society*, 357(5):1953–1962, 2005. doi:10.1090/S0002-9947-04-03555-X.

[14] Chris Godsil and Gordon F. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer, New York, 2001. doi:10.1007/978-1-4613-0163-9.

[15] Frank Harary. *Graph Theory*. Addison-Wesley, Reading, MA, 1969.

[16] Bernt Tore Jensen, Xiuping Su, and Rupert Wei Tze Yu. Rigid representations of a double quiver of type A, and Richardson elements in seaweed Lie algebras. *Bulletin of the London Mathematical Society*, 41(1):1–15, 2009. doi:10.1112/blms/bdn087.

[17] Christian Kassel, Alain Lascoux, and Christophe Reutenauer. The singular locus of a Schubert variety. *Journal of Algebra*, 269(1):74–108, 2003. doi:10.1016/S0021-8693(03)00014-0.

[18] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Inventiones Mathematicae*, 53(2):165–184, 1979. doi:10.1007/BF01390031.

[19] David Kazhdan and George Lusztig. Schubert varieties and Poincaré duality. In Robert Osserman and Alan Weinstein, editors, *Geometry of the Laplace Operator*, volume 36 of *Proceedings of Symposia in Pure Mathematics*, pages 185–203. American Mathematical Society, Providence, RI, 1980. doi:10.1090/pspum/036/573434.

[20] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in SL(n)/b. *Proceedings of the Indian Academy of Sciences. Mathematical Sciences*, 100:45–52, 1990. doi:10.1007/BF02881113.

[21] V. Lakshmibai and C. S. Seshadri. Singular locus of a Schubert variety. *Bulletin of the American Mathematical Society*, 11(2):363–366, 1984. doi:10.1090/S0273-0979-1984-15309-6.

[22] Erez Lapid. A greedy open-orbit criterion for solvable algebraic group actions, with applications to Lusztig's nilpotent varieties, 2026. arXiv:2608.14413, doi:10.48550/arXiv.2608.14413.

[23] Erez Lapid and Alberto Mínguez. Geometric conditions for □-irreducibility of certain representations of the general linear group over a non-archimedean local field. *Advances in Mathematics*, 339:113–190, 2018. doi:10.1016/j.aim.2018.09.027.

[24] Erez Lapid and Alberto Mínguez. Conjectures and results about parabolic induction of representations of $\mathrm{GL}_n(F)$. *Inventiones Mathematicae*, 222(3):695–747, 2020. doi:10.1007/s00222-020-00982-7.

[25] Erez Lapid and Alberto Mínguez. A binary operation on irreducible components of Lusztig's nilpotent varieties II: applications and conjectures for representations of GL_n over a non-archimedean local field. *Pure and Applied Mathematics Quarterly*, 21(2):813–863, 2025. doi:10.4310/PAMQ.241205005734.

[26] Laurent Manivel. Le lieu singulier des variétés de Schubert. *International Mathematics Research Notices*, (16):849–871, 2001. doi:10.1155/S1073792801000423.

[27] Emmanuel Tsukerman and Lauren Williams. Bruhat interval polytopes. *Advances in Mathematics*, 285:766–810, 2015. arXiv:1406.5202, doi:10.1016/j.aim.2015.07.030.

[28] Alexander Woo and Alexander Yong. When is a Schubert variety Gorenstein? *Advances in Mathematics*, 207(1):205–220, 2006. doi:10.1016/j.aim.2005.11.010.

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