S.T. Yau High School Science Award

Research Report

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Title of Research Report

Smoothness Characterizations and Duality of the Type II Theta Correspondence and Basic Fourier–Jacobi Models

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SMOOTHNESS CHARACTERIZATIONS AND DUALITY OF THE TYPE II THETA CORRESPONDENCE AND BASIC FOURIER–JACOBI MODELS

BRIAN SUN AND SEABERT MAO

Abstract. We study the smoothness of two families of Borel-orbit closures. The first consists of matrix Schubert varieties in $\mathrm{Mat}_{n\times m}$. The second consists of orbit closures for the diagonal action of the upper-triangular Borel subgroup $B \subseteq \mathrm{GL}(V)$ on $\mathrm{FL}(V) \times V$, where $V = \mathbb{C}^n$.

For matrix Schubert varieties, we prove that smoothness is equivalent to the absence of a contiguous 2×2 block of equal positive entries in the southwest rank table of the indexing partial permutation matrix. The proof also shows that every smooth matrix Schubert variety is a coordinate linear subspace. We enumerate these varieties both in total and by rank; the rank-refined counts are Catalan triangle numbers.

The B-orbits in $\mathrm{FL}(V) \times V$ are indexed by colored permutations. We give a complete smoothness criterion involving two ordinary permutation patterns and two colored patterns, together with two exceptional extensions of the final colored pattern.

When $m = n$, the Borel orbits in the two ambient spaces are naturally in bijection, and hence so are their closures. In the framework of relative Langlands duality, this is the expected correspondence between the type II theta-correspondence model and the basic Fourier–Jacobi model. We show that the bijection does not preserve smoothness in general and characterize the failures in one direction.

Keywords. spherical varieties, smoothness criteria, pattern avoidance, matrix Schubert varieties, type II theta-correspondence model, basic Fourier–Jacobi model, relative Langlands duality

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GPT-5.6 Sol and Claude Opus 4.7/4.8 were used around once or twice per week, totaling six weeks throughout our research. It assisted by polishing language, identifying potential mathematical errors, organizing data, and drafting tikz diagrams which were manually edited and verified.

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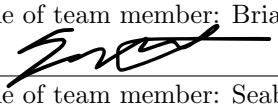
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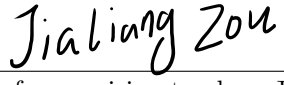
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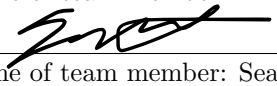
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The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

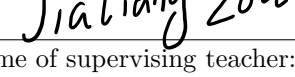
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1. Introduction

The theorem of Lakshmibai and Sandhya gives a classical connection between the geometry of Schubert varieties and permutation pattern avoidance. Let $B \subseteq \mathrm{GL}_n(\mathbb{C})$ be the Borel subgroup of invertible upper-triangular matrices, and let X_w be the Schubert variety in the complete flag variety $\mathrm{GL}_n(\mathbb{C})/B$ indexed by $w \in S_n$. A permutation $w \in S_n$ is said to contain a pattern $p \in S_k$ if some subsequence of w has the same relative order as p; otherwise, w avoids p. Lakshmibai and Sandhya proved the following criterion [LS90]:

$$X_w \text{ is smooth} \iff w \text{ avoids 3412 and 4231.}$$

Thus the smoothness of a type A Schubert variety is detected by avoiding two finite patterns. This theorem is a prototype for the broader use of pattern avoidance to study singularities of orbit closures. For the other classical types, Billey gave pattern-avoidance criteria for rational smoothness [Bil98], and Billey and Postnikov later found a uniform formulation for all types using patterns in root subsystems [BP05].

The smoothness of Borel-orbit closures in symmetric varieties has also been studied. McGovern and Trapa obtained explicit pattern-avoidance criteria for several classical symmetric pairs [MT09], while Hultman gave a Bruhat-graph criterion for rational smoothness [Hul12].

Flag varieties and symmetric varieties both belong to the broader class of spherical varieties. Recall that a normal G-variety X is *spherical* if a Borel subgroup $B \subseteq G$ has only finitely many orbits on X [Vin86, Bri86]. It is therefore natural to seek combinatorial smoothness criteria for B-orbit closures in other families of spherical varieties.

In this paper, we obtain such criteria for two families. The first consists of matrix Schubert varieties, which are indexed by partial permutation matrices rather than permutations. For the second, the same Borel subgroup B acts diagonally on $\mathrm{FL}_n \times \mathbb{C}^n$ by

$$b \cdot (F_\bullet, v) = (bF_\bullet, bv).$$

We study the closures of these B-orbits, which are indexed by permutations whose entries carry additional color data. Our main results give complete smoothness criteria for both families: a local rank-table pattern for matrix Schubert varieties and a combination of ordinary and colored permutation patterns for the second case.

We next state our main results for the two models.

1.1. Matrix Schubert Varieties.

We first consider matrix Schubert varieties, which are closed affine subvarieties of $\mathrm{Mat}_{n\times m}$ defined by rank conditions on submatrices. Introduced by Fulton in his study of degeneracy loci and Schubert polynomials [Ful92], matrix Schubert varieties have since become important objects in algebraic geometry, algebraic combinatorics, and commutative algebra [KM05, MS05].

We work over $\mathbb{C}$. Let

$$X = \mathrm{Mat}_{n\times m}$$

be the space of complex $n \times m$ matrices. Let

$$G = \mathrm{GL}_n \times \mathrm{GL}_m$$

act on X by left and right multiplication:

$$(g, h) \cdot A = gAh^{-1},$$

and let $B = B_1 \times B_2$, where $B_1 \subset \mathrm{GL}_n$ and $B_2 \subset \mathrm{GL}_m$ are the upper-triangular Borel subgroups. The G-variety X is the space underlying the type II theta-correspondence model. Its B-orbits are indexed by partial permutation matrices, or equivalently by partial matchings

$$\sigma = (I, J, \sigma) \in \mathrm{PM}(m, n),$$

where $I \subseteq \{1, \ldots, m\}$, $J \subseteq \{1, \ldots, n\}$, and $\sigma : I \to J$ is a bijection (see Lemma 2.4). By a slight abuse of notation, we use σ to denote both the partial matching $(I, J, \sigma) \in \mathrm{PM}(m, n)$ and its underlying bijection $\sigma : I \to J$. We write M_σ for the corresponding partial permutation matrix, with a 1 in row $\sigma(i)$ and column i for each $i \in I$, and zeros elsewhere. For $\sigma \in \mathrm{PM}(m, n)$, let $\mathcal{O}_\sigma$ denote the B-orbit of M_σ in X. Its closure $\overline{\mathcal{O}_\sigma}$ is the matrix Schubert variety associated to σ.

We now introduce the notion of a southwest rank table for a partial permutation matrix M_σ.

Definition 1.1. For a partial permutation $\sigma \in \mathrm{PM}(m, n)$, the *southwest rank table* $\mathrm{rk}(M_\sigma)$ of the corresponding partial permutation matrix M_σ is

$$\mathrm{rk}(M_\sigma)[r, c] = \#\{\, h \in I : h \leq c \text{ and } \sigma(h) \geq r \,\}.$$

Equivalently, the entry $\mathrm{rk}(M_\sigma)[r, c]$ is the number of 1's in M_σ weakly southwest of the position (r, c).

We say that the southwest rank table contains a contiguous 2×2 block of equal, nonzero entries if there exist indices r and c such that

$$\mathrm{rk}(M_\sigma)[r, c] = \mathrm{rk}(M_\sigma)[r + 1, c] = \mathrm{rk}(M_\sigma)[r, c + 1] = \mathrm{rk}(M_\sigma)[r + 1, c + 1] \neq 0.$$

The main result of Section 2 is the following smoothness criterion.

Theorem 1.2. *The matrix Schubert variety $\overline{\mathcal{O}_\sigma}$ for $\sigma \in \mathrm{PM}(m, n)$ is smooth if and only if the southwest rank table of M_σ contains no contiguous 2×2 block of equal, nonzero entries.*

Thus smoothness of a matrix Schubert variety is determined by a local pattern-avoidance condition in its southwest rank table. The following example illustrates the criterion.

Example 1.3. Consider the two partial permutation matrices

$$M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{\sigma'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Their southwest rank tables are

$$\mathrm{rk}(M_\sigma) = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{rk}(M_{\sigma'}) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

The southwest rank table $\mathrm{rk}(M_\sigma)$ avoids a contiguous 2×2 block of equal, nonzero entries. Thus, by Theorem 1.2, the matrix Schubert variety $\overline{\mathcal{O}_\sigma}$ is smooth.

The southwest rank table $\mathrm{rk}(M_{\sigma'})$ contains a contiguous 2×2 block of equal, nonzero entries. Thus, by Theorem 1.2, the matrix Schubert variety $\overline{\mathcal{O}_{\sigma'}}$ is singular.

The proof of Theorem 1.2 has two main steps. First, we characterize the matrix Schubert varieties that are linear. We show that $\overline{\mathcal{O}_\sigma}$ is linear precisely when the southwest rank table of M_σ avoids a contiguous 2×2 block of equal, nonzero entries. Second, we prove that every smooth matrix Schubert variety is linear. This is done by computing the tangent space dimension at the empty partial matching and comparing it with the dimension forced by the rank conditions. Combining these two results gives Theorem 1.2. An important consequence of the proof is that

smoothness is very rigid in this setting: every smooth matrix Schubert variety is a coordinate linear subspace.

We also use the smoothness criterion to count smooth matrix Schubert varieties. For fixed dimensions m and n, we show that the total number of smooth matrix Schubert varieties in $\mathrm{Mat}_{n\times m}$ is $\binom{m+n}{m}$. We further refine this count by the rank of the indexing partial permutation matrix M_σ. For $1 \le k \le \min(m,n)$, the number of smooth matrix Schubert varieties of rank k is given by the Catalan triangle number $C(n+m-k,k)$ (see Theorem 2.29). In particular, the number of smooth full-rank matrix Schubert varieties is $C(\max(m,n),\min(m,n))$. When $m=n$, this specializes to the usual Catalan number.

1.2. **Borel Orbit Closures in $\mathrm{FL}(V)\times V$.** Let $V=\mathbb{C}^n$, let $B\subseteq \mathrm{GL}(V)$ be the Borel subgroup of invertible upper-triangular matrices, and set

$$Y = \mathrm{FL}(V)\times V.$$

The group B acts diagonally on Y by

$$b\cdot(F_\bullet, v) = (bF_\bullet, bv).$$

The G-variety Y is the space underlying the basic Fourier–Jacobi model. This orbit problem belongs to finite-dimensional mirabolic geometry. Its orbits are finite in number and are parametrized by decorated permutations [Sol72, MWZ99, Tra09].

More precisely, the B-orbits in Y are indexed by pairs

$$(w,\alpha)\in \mathrm{RB}(n),$$

where $w\in S_n$, and $\alpha\subseteq[n]$ is an antichain in the poset $[n]_w$ defined by

$$i <_w j \quad\Longleftrightarrow\quad i<j \text{ and } w(i)<w(j)$$

(see Lemma 3.4). Equivalently, if $i,j\in\alpha$ with $i<j$, then $w(i)>w(j)$. We visualize elements of $\mathrm{RB}(n)$ as permutations whose entries at position i are colored red if $i\in\alpha$ and blue if $i\notin\alpha$. Thus the antichain condition says precisely that the red entries form a decreasing subsequence.

Example 1.4. The antichain $(w,\alpha)=(3412,\{1,3\})$ is visualized as 3412.

The orbit corresponding to $(w,\alpha)\in\mathrm{RB}(n)$, denoted by $\mathcal{O}_{(w,\alpha)}$, has representative

$$(w\cdot E_\bullet, v_\alpha), \qquad v_\alpha=\sum_{a\in\alpha} e_{w(a)},$$

where $\{e_1,\dots,e_n\}$ is the standard basis of V and $E_\bullet$ is the corresponding standard flag. The closures of these orbits, denoted by $\overline{\mathcal{O}_{(w,\alpha)}}$, are the objects whose smoothness we wish to characterize.

To state the criterion, we need the notion of colored pattern avoidance.

Definition 1.5. Let $(w,\alpha)\in\mathrm{RB}(n)$ be a colored permutation and let (p,γ) be a colored pattern of size k. We say (w,α) *contains* (p,γ) if there exist indices $1\le i_1<\cdots<i_k\le n$ such that

$$w(i_a)>w(i_b) \iff p(a)>p(b)$$

and $i_j\in\alpha \iff j\in\gamma$ for each $j\in[k]$. Otherwise (w,α) *avoids* (p,γ).

The main result of Section 3 is the following smoothness criterion.

Theorem 1.6. *Let $(w,\alpha)\in\mathrm{RB}(n)$. The orbit closure $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular if and only if (w,α) satisfies at least one of the following conditions:*

1. *the permutation w contains the pattern 3412, regardless of color;*
2. *the permutation w contains the pattern 4231, regardless of color;*
3. *the colored permutation (w,α) contains the pattern 321;*
4. *the colored permutation (w,α) contains the pattern 321, and this pattern is not contained in an occurrence of either 3421 or 4312.*

Example 1.7. Table 1 shows seven colored permutations, whether the corresponding orbit closure is smooth or singular, and how Theorem 1.6 applies in each case.

TABLE 1. Smoothness for various colored permutations in the basic Fourier–Jacobi model.

(w, α)	$\overline{\mathcal{O}_{(w,\alpha)}}$	Reason for Being Smooth or Singular
4321	Smooth	Avoids $3412, 4231$; no occurrence of 321 or 321.
34125	Singular	Contains 3412 at positions 1–4.
153426	Singular	Contains 4231 at positions 2–5.
4321	Singular	Contains 321 at positions $1, 2, 3$.
4321	Singular	Contains 321 at $(1, 2, 4)$ and $(1, 3, 4)$, neither in an occurrence of 3421 or 4312.
3421	Smooth	Contains 321 at $(1, 3, 4)$ (as its only singular pattern), in an occurrence of 3421.
4312	Smooth	Contains 321 at $(1, 2, 3)$ (as its only singular pattern), in an occurrence of 4312.

The proof of Theorem 1.6 has two main steps. First, we compute the dimension of the orbit closures $\overline{\mathcal{O}_{(w,\alpha)}} \subseteq \mathrm{FL}(V) \times V$ in terms of the indexing antichain (w, α). Then, we compute the dimension of the tangent space to $\overline{\mathcal{O}_{(w,\alpha)}}$ at the standard base point $(E_\bullet, 0)$. Comparing the two dimensions yields the result.

1.3. **Speculation on the Duality Between the Two Models.** When $m = n$, the Borel orbits in the two above cases are naturally in bijection:

$$\Psi : \mathrm{PM}(n, n) \xrightarrow{\sim} \mathrm{RB}(n).$$

Finkelberg, Ginzburg, and Travkin construct this bijection through the intermediate combinatorics of $n \times n$ mazes [FGT25, §4.2]. In their framework, Ψ is the bijection expected from relative Langlands duality [BZSV24] between the type II theta-correspondence model and the basic Fourier–Jacobi model [FGT25]. A natural first test of this duality is whether Ψ preserves the smoothness of orbit closures.

Our results show that smoothness is not preserved in general. Nevertheless, we completely characterize the failures in one direction: a singular Fourier–Jacobi orbit closure corresponds to a singular matrix Schubert variety unless its only obstruction to smoothness is an occurrence of 4231 colored as 4231.

The rest of the paper is organized as follows. Section 2 proves the rank-table criterion for matrix Schubert varieties, and Subsection 2.4 uses it to enumerate the smooth varieties for fixed matrix dimensions, both in total and by rank. Section 3 proves the colored-pattern criterion for the basic Fourier–Jacobi model. Finally, Section 4 compares smoothness under the correspondence between the two models and discusses directions for future work.

2. SMOOTHNESS CHARACTERIZATION FOR MATRIX SCHUBERT VARIETIES

In this section, we prove a rank-table characterization of smooth matrix Schubert varieties. We first build on [MQYZ25, Lemma D.9] to characterize the linear matrix Schubert varieties in terms of their rank tables. We then show that smoothness forces linearity, which gives the complete smoothness criterion stated in Theorem 1.2.

Throughout this section, we work over $\mathbb{C}$ so that smoothness is understood in the usual sense for complex algebraic varieties.

2.1. **Preliminaries for Matrix Schubert Varieties.** We start by covering the necessary background on matrix Schubert varieties and the related combinatorial notions. We follow [MS05, Chapter 15].

Remark 2.1. We note that our notion of matrix Schubert varieties differs from the convention used in [MS05, §15]. In [MS05, §15], matrix Schubert varieties are defined as Borel orbit closures

for the action of the lower triangular Borel subgroup on the left and the upper triangular Borel subgroup on the right. In contrast, throughout this paper we use the action of the upper triangular Borel subgroups on both sides. These two conventions are equivalent up to the involution given by reversing the order of the rows, and thus the corresponding classes of varieties are isomorphic.

Definition 2.2. (1) A *partial matching* between the sets $\{1,\dots,m\}$ and $\{1,\dots,n\}$ is a triple $\sigma = (I, J, \sigma)$, where $I \subseteq \{1,\dots,m\}$, $J \subseteq \{1,\dots,n\}$, and $\sigma : I \to J$ is a bijection.
(2) Let $\mathrm{PM}(m,n)$ denote the set of all partial matchings between $\{1,\dots,m\}$ and $\{1,\dots,n\}$.
(3) The *partial permutation matrix* M_σ is the matrix that is 1 on set of indices $\{(\sigma(i), i) \mid i \in I\}$ and 0 elsewhere. The position $(\sigma(i), i)$ denotes row $\sigma(i)$ and column i.

Example 2.3. Consider the partial matching $\sigma = (I, J, \sigma)$, where $I = \{1, 3\}$, $J = \{3, 4\}$, and

$$\sigma(1) = 4, \qquad \sigma(3) = 3.$$

We denote σ in one-line notation as $[(1, 4), (3, 3)]$. The partial permutation matrix M_σ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We use σ and the partial permutation matrix M_σ interchangeably, as is common for permutation matrices. Note the convention: for partial permutations in one-line notation, each ordered pair (i, j) denotes a 1 in M_σ in row j and column i. For all other references to matrix entries in this paper, the notation $M_\sigma[r, c]$ refers to the entry in row r and column c.

Lemma 2.4. [MS05, Proposition 15.27] *There is a bijection between* $\mathrm{PM}(m,n)$ *and the set of* $B = B_1 \times B_2$*-orbits on* $X = \mathrm{Mat}_{n\times m}$*. In particular, each orbit contains a unique partial permutation matrix as its representative element.*

For each $\sigma \in \mathrm{PM}(m,n)$, let $\mathcal{O}_\sigma$ denote the corresponding B-orbit. Then, we have a stratification

$$X = \bigsqcup_{\sigma\in\mathrm{PM}(m,n)} \mathcal{O}_\sigma.$$

The closures of the B-orbits $\mathcal{O}_\sigma$, denoted by $\overline{\mathcal{O}_\sigma}$, are the matrix Schubert varieties. Our goal is to characterize when these matrix Schubert varieties are smooth. Note that $\overline{\mathcal{O}_\sigma}$ is a union of B-orbits. This gives a partial order on $\mathrm{PM}(m,n)$, defined by

$$\sigma' \le \sigma \quad \text{if and only if} \quad \mathcal{O}_{\sigma'} \subseteq \overline{\mathcal{O}_\sigma}.$$

We refer to this as the Bruhat order on $\mathrm{PM}(m,n)$. The Bruhat order can be described using the southwest rank table defined in Definition 1.1.

Lemma 2.5. [MS05, Lemma 15.19] *Let* $\sigma \in \mathrm{PM}(m,n)$*. In Bruhat order, we have* $\sigma' \le \sigma$ *if and only if*

$$\mathrm{rk}(M_{\sigma'})[r, c] \le \mathrm{rk}(M_\sigma)[r, c]$$

for all $1 \le r \le n$ *and* $1 \le c \le m$*.*

2.2. **Characterization of Linear Matrix Schubert Varieties.** To index and construct all $B_1 \times B_2$-stable linear subspaces of $\mathrm{Mat}_{n\times m}$, we introduce some definitions.

Definition 2.6. [MQYZ25, Definition D.8] Let

$$\mathrm{PP}(m,n) := \{(I, J) \mid I \subseteq \{1,\dots,m\},\ J \subseteq \{1,\dots,n\},\ |I| = |J|\}$$

denote the pairs of subsets of $\{1,\dots,m\}$ and $\{1,\dots,n\}$ having the same cardinality.

Definition 2.7. [MQYZ25] For $\mu = (I, J) \in \mathrm{PP}(m, n)$ with

$$I = \{i_1 < i_2 < \cdots < i_k\}, \qquad J = \{j_1 < j_2 < \cdots < j_k\},$$

let $\overline{L}_1^\mu \subseteq \mathrm{Mat}_{n\times m}$ be the subspace defined by the block matrix configuration in Figure 1 below. The shaded region indicates the free entries in $\overline{L}_1^\mu$, and the filled circles mark its corners; the unshaded entries below the staircase are all 0.

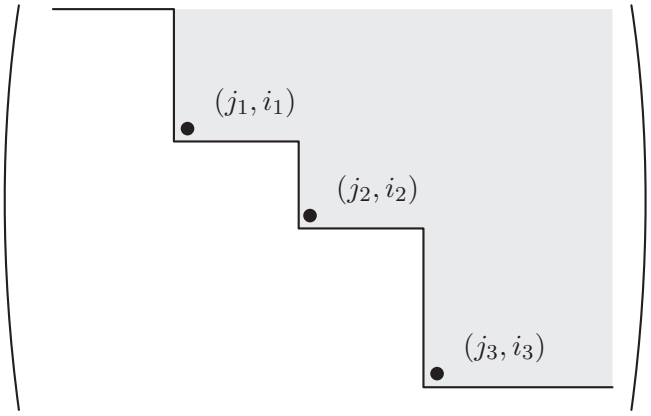


FIGURE 1. Block matrix configuration of $\overline{L}_1^\mu$

Lemma 2.8. *The $B_1 \times B_2$-stable linear subspaces of $\mathrm{Mat}_{n\times m}$ are exactly the subspaces $\overline{L}_1^\mu$ from Definition 2.7.*

Proof. Recall that the action of $B_1 \times B_2$ on $\mathrm{Mat}_{n\times m}$ is

$$(b_1, b_2) \cdot X = b_1 X b_2^{-1},$$

where $B_1 \subseteq \mathrm{GL}_n$ and $B_2 \subseteq \mathrm{GL}_m$ are the upper triangular Borel subgroups.

First, let $\mu = (I, J) \in \mathrm{PP}(m, n)$, where

$$I = \{i_1 < \cdots < i_k\}, \qquad J = \{j_1 < \cdots < j_k\}.$$

By Definition 2.7, the subspace $\overline{L}_1^\mu$ consists of certain entries which are free and others which always vanish. Therefore $\overline{L}_1^\mu$ is closed under scaling and addition and hence is a linear subspace of $\mathrm{Mat}_{n\times m}$.

We claim that $\overline{L}_1^\mu$ is $B_1 \times B_2$-stable. Left multiplication by b_1 pushes nonzero entries north and right multiplication by b_2^{-1} pushes nonzero entries east. Both operations preserve the northeast staircase region defining $\overline{L}_1^\mu$ from Figure 1. Hence $\overline{L}_1^\mu$ is $B_1 \times B_2$-stable.

Conversely, let $W \subseteq \mathrm{Mat}_{n\times m}$ be a $B_1 \times B_2$-stable linear subspace. We first show that W is spanned by matrix units. Let $X = (x_{p,q}) \in W$, and choose an arbitrary entry $x_{r,c} \neq 0$. We first isolate row r. Choose a scalar $t \neq 0, 1$, and let

$$D_r(t) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{r-1}, t, 1, \ldots, 1) \in B_1,$$

where the entry t occurs in position r. Since W is stable under left multiplication by B_1, both $D_r(t)X$ and X lie in W. Hence, because W is linear,

$$\frac{D_r(t)X - X}{t - 1} \in W.$$

This matrix has its r-th row equal to the r-th row of X and all other rows equal to zero.

Let this matrix be Y. Its only possible nonzero entries are in row r, and its (r, c)-entry is still $x_{r,c} \neq 0$. We next isolate column c. Choose a scalar $u \neq 0, 1$, and let

$$D_c(u) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{c-1}, u, 1, \ldots, 1) \in B_2,$$

where the entry u occurs in position c. Since W is stable under right multiplication by B_2^{-1}, both $YD_c(u)^{-1}$ and Y lie in W. Therefore,

$$\frac{u}{1-u}\big(YD_c(u)^{-1} - Y\big) \in W.$$

This matrix has a single nonzero entry, namely $x_{r,c}$ in position (r, c). Since $x_{r,c} \neq 0$ and W is a linear subspace, it follows that $E_{r,c} \in W$.

Thus, for every $X \in W$, each nonzero entry of X determines a matrix unit in W at the same position. Since X is the sum of its entries times the corresponding matrix units, it follows that W is spanned by matrix units. Stability of W under left multiplication by B_1 and right multiplication by B_2^{-1} implies that these matrix units are closed under moving upward in rows and to the right in columns. Therefore, the matrix units contained in W form an upper-right staircase region.

Hence W is exactly the subspace of matrices whose entries are free in this staircase region and vanish outside it. By Definition 2.7, this means that $W = \overline{L}_1^{\mu}$ for some $\mu \in \mathrm{PP}(m, n)$. □

As explained in [MQYZ25], because each $\overline{L}_1^{\mu}$ is $B_1 \times B_2$-stable and an irreducible subvariety, there exists a unique $\sigma \in \mathrm{PM}(m, n)$ such that

$$\overline{L}_1^{\mu} = \overline{\mathcal{O}_\sigma}.$$

Definition 2.9. Let $\Phi : \mathrm{PP}(m, n) \to \mathrm{PM}(m, n)$ map $\mu \in \mathrm{PP}(m, n)$ to the $\sigma \in \mathrm{PM}(m, n)$ such that $\overline{L}_1^{\mu} = \overline{\mathcal{O}_\sigma}$.

An algorithm for $\Phi(\mu)$ is given in [MQYZ25, Lemma D.9] and is restated below as it is central to our later proofs.

Lemma 2.10. [MQYZ25, Lemma D.9] *Given $\mu = (I, J) \in \mathrm{PP}(m, n)$ with*

$$I = \{i_1 < i_2 < \cdots < i_k\} \qquad \text{and} \qquad J = \{j_1 < j_2 < \cdots < j_k\},$$

the partial matching $\sigma \in \mathrm{PM}(m, n)$ such that $\Phi(\mu) = \sigma$ can be determined by the following algorithm.

(1) Initialize an empty partial matching σ and an empty stack S.
(2) For each position $p = 1, \ldots, k$:
  (a) Push the integers

$$j_{p-1} + 1,\ j_{p-1} + 2,\ \ldots,\ j_p - 1,\ j_p$$

  onto the stack S one by one, where $j_0 = 0$ for the first iteration.
  (b) For each integer $x = i_p, \ldots, i_{p+1} - 1$, where $i_{k+1} = m + 1$ for the last iteration:
    - *If the stack S is nonempty, pop an element y from S and add the pair (x, y) to the partial matching σ.*
    - *If the stack S is empty, break out of the inner loop and proceed to the next position.*
(3) Return the constructed partial matching σ.

Note that σ is empty if μ is empty, as determined by the algorithm.

For our purposes, we will use the following less-technical rephrasing of the algorithm.

Corollary 2.11. Given $\mu \in \mathrm{PP}(m, n)$ with

$$I = \{i_1 < i_2 < \cdots < i_k\} \qquad \text{and} \qquad J = \{j_1 < j_2 < \cdots < j_k\},$$

the following procedure is an equivalent algorithm for the function determining the partial matching $\sigma \in \mathrm{PM}(m, n)$ such that $\Phi(\mu) = \sigma$:

(1) Start with the partial permutation matrix M_σ as all 0's. For each $1 \leq p \leq k$, set $M_\sigma[j_p, i_p]$ to 1. We call these the *base points*.

(2) Perform the following procedure on the base points from left to right, i.e., (j_1, i_1) first. For each base point (j_p, i_p), move one column right to column $i_p + 1$. Determine the maximum row index less than j_p whose row is empty, and set that entry in column i_p+1 equal to 1. Repeat this algorithm for all columns $i_p + 1, i_p + 2, \ldots, i_{p+1} - 1$, moving one column right each time and taking the largest available row index less than j_p. We define $i_{k+1} = m + 1$ for the last iteration.
(3) If we run out of empty rows above row j_p while proceeding across columns $i_p + 1, i_p + 2, \ldots, i_{p+1} - 1$, then skip to the next base point (j_{p+1}, i_{p+1}) and repeat Step (2) from there.

The resulting partial permutation matrix then gives the partial permutation $\sigma = \Phi(\mu)$.

Example 2.12. Let $m = 8$ and $n = 7$, and let $\mu \in \mathrm{PP}(8, 7)$ be given by

$$\mu = (\{1, 4, 8\}, \{4, 6, 7\}).$$

Equivalently, the matrix associated with μ is

$$M_\mu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Applying the procedure in Corollary 2.11, we obtain the partial permutation matrix

$$M_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$\Phi(\mu) = \sigma = [(1, 4), (2, 3), (3, 2), (4, 6), (5, 5), (6, 1), (8, 7)].$$

The rephrasing in Corollary 2.11 helps us to characterize all linear matrix Schubert varieties $\overline{\mathcal{O}_\sigma}$ using southwest rank tables.

Lemma 2.13. *For a partial matching $\sigma \in \mathrm{PM}(m, n)$, if there exists an element $\mu \in \mathrm{PP}(m, n)$ such that $\Phi(\mu) = \sigma$, then σ avoids a contiguous 2×2 block of equal, nonzero entries in its southwest rank table.*

Proof. Let $\Phi(\mu) = \sigma$, and suppose for the sake of contradiction that the southwest rank table of M_σ contains a contiguous 2×2 block of equal, nonzero entries:

$$\mathrm{rk}(M_\sigma)[r, c] = \mathrm{rk}(M_\sigma)[r + 1, c] = \mathrm{rk}(M_\sigma)[r, c + 1] = \mathrm{rk}(M_\sigma)[r + 1, c + 1] \neq 0. \tag{2.1}$$

Since $\mathrm{rk}(M_\sigma)[r, c]$ counts the number of 1's in the southwest submatrix with northeast corner (r, c), Equation (2.1) implies the following about the arrangement of ones in M_σ:

I. there is no 1 in row r in any column with index $\leq c + 1$;
II. there is no 1 in column $c + 1$ in any row with index $\geq r + 1$;
III. there is at least one 1 in the rectangle

$$S := \{(a, b) \mid a \geq r + 1,\ b \leq c\}.$$

Choose the 1 in S with maximal column index v, and denote its position as (u, v). Thus

$$u \geq r+1, \qquad v \leq c,$$

and there is no 1 of M_σ in S in any column larger than v.

From the algorithm for Φ in Corollary 2.11, the entry (u, v) either is a base point (j_p, i_p), in which case $v = i_p$, or is produced from the unique preceding base point (j_p, i_p), in which case $i_p < v < i_{p+1}$. In either case, after the algorithm places the 1 at (u, v), exactly one of the following occurs:

Case 1: The algorithm continues with the same base point into column $v+1$.

Then, by Step (2), it places a 1 in column $v+1$ in the maximum available row strictly less than j_p. Since row r is empty in every column $\leq c+1$ by (I.), and since $r < j_p$, row r is still available at this stage. Hence, the algorithm places a 1 in column $v+1$ in some row $u' \geq r$.

If $u' = r$, then row r contains a 1 in a column with index $\leq c+1$, contradicting (I.). Thus $u' \geq r+1$. If $v+1 \leq c$, then $(u', v+1) \in S$, contradicting the maximality of v. If $v+1 = c+1$, then column $c+1$ contains a 1 in a row with index $\geq r+1$, contradicting (II.). Therefore, Case 1 is impossible.

Case 2: The algorithm does not continue with the same base point after placing the 1 at (u, v).

By Steps (2) and (3), this can only happen if there are no available rows remaining in column $v+1$ strictly less than j_p or there is a base point in column $v+1$. However, row r is strictly less than j_p and is still available in column $v+1 \leq c+1$, again by (I.). This is a contradiction, so the only other possible case is that we have a base point in column $v+1$.

Since the base points satisfy

$$j_1 < j_2 < \cdots,$$

we have

$$j_{p+1} > j_p \geq u \geq r+1.$$

Thus, column $v+1$ contains a 1 in a row $\geq r+1$.

If $v+1 \leq c$, then this 1 lies in S, again contradicting the maximality of v. If $v+1 = c+1$, then column $c+1$ contains a 1 in a row $\geq r+1$, contradicting (II.). Therefore, Case 2 is also impossible.

Both cases lead to contradictions. Therefore M_σ cannot contain a contiguous 2×2 block of equal, nonzero entries in its southwest rank table. □

We now show the reverse direction. First, we introduce some terminology and prove three structural lemmas. Assume that the southwest rank table of M_σ contains no contiguous 2×2 block of equal, nonzero entries. We prove by construction that there exists some $\mu \in \mathrm{PP}(m, n)$ such that $\Phi(\mu) = \sigma$.

If M_σ has no ones, then σ is the empty partial permutation. In this case, take $\mu = (\emptyset, \emptyset) \in \mathrm{PP}(m, n)$. The algorithm from Corollary 2.11 then produces the empty partial permutation, so $\Phi(\mu) = \sigma$.

We now assume that M_σ has $t \geq 1$ entries equal to 1. List the positions of these 1's from left to right as

$$(r_1, c_1), \ldots, (r_t, c_t),$$

so that

$$c_1 < c_2 < \cdots < c_t.$$

Define indices

$$1 = b_1 < b_2 < \cdots < b_k \leq t$$

recursively as follows. Set $b_1 = 1$. For each b_a, let b_{a+1} be the minimal index greater than b_a such that

$$r_{b_a} < r_{b_{a+1}}. \tag{2.2}$$

If no such index exists, stop the recursion.

These b_i will index our base points for the algorithm defined in Corollary 2.11. We call the ones in M_σ with positions indexed by

$$b_a, b_a + 1, \ldots, b_{a+1} - 1,$$

the "run" from the base point (r_{b_a}, c_{b_a}).

Now construct

$$\mu = (I, J) \in \mathrm{PP}(m, n)$$

by setting

$$I = \{c_{b_1} < \cdots < c_{b_k}\}, \qquad J = \{r_{b_1} < \cdots < r_{b_k}\}. \tag{2.3}$$

We claim that

$$\Phi(\mu) = \sigma.$$

We prove three structural claims about the arrangement of ones in M_σ.

Lemma 2.14. *Let σ avoid a contiguous 2×2 block of equal, nonzero entries in its southwest rank table. Let p be an index such that $r_p > r_{p+1}$. Then,*

$$c_{p+1} = c_p + 1.$$

Proof. Suppose for the sake of contradiction that $c_{p+1} > c_p + 1$. Since the unique 1 in row r_{p+1} lies in column $c_{p+1} > c_p + 1$, row r_{p+1} contains no 1 in any column $\leq c_p + 1$. Moreover, since by definition there is no 1 in any column strictly between c_p and c_{p+1}, column $c_p + 1$ contains no 1 in any row. Thus,

$$\mathrm{rk}(M_\sigma)[r_{p+1}, c_p] = \mathrm{rk}(M_\sigma)[r_{p+1} + 1, c_p] = \mathrm{rk}(M_\sigma)[r_{p+1}, c_p + 1] = \mathrm{rk}(M_\sigma)[r_{p+1} + 1, c_p + 1].$$

Additionally, we have that (r_p, c_p) lies in the southwest rectangle

$$\{(u, v) \mid u \geq r_{p+1} + 1,\ v \leq c_p\},$$

because $r_p > r_{p+1}$. Hence,

$$\mathrm{rk}(M_\sigma)[r_{p+1} + 1, c_p] \neq 0.$$

Therefore,

$$\mathrm{rk}(M_\sigma)[r_{p+1}, c_p] = \mathrm{rk}(M_\sigma)[r_{p+1} + 1, c_p] = \mathrm{rk}(M_\sigma)[r_{p+1}, c_p + 1] = \mathrm{rk}(M_\sigma)[r_{p+1} + 1, c_p + 1] \neq 0,$$

which is a contiguous 2×2 block of equal, nonzero entries in the southwest rank table. Thus, we reach a contradiction. □

Lemma 2.15. *Let σ avoid a contiguous 2×2 block of equal, nonzero entries in its southwest rank table. Fix a run beginning at the base point with index b_a. Suppose that p and $p + 1$ both lie before the next base point, i.e.*

$$b_a \leq p < b_{a+1} - 1,$$

or, if $a = k$,

$$b_a \leq p < t.$$

Then,

$$r_p > r_{p+1}.$$

Proof. This lemma states that the runs defined by the base points in Equation (2.2) are decreasing in row indices.

For the sake of contradiction, suppose that some row index increases within the run beginning at b_a. Choose the minimal p such that

$$b_a \leq p < b_{a+1} - 1$$

or, if $a = k$,

$$b_a \leq p < t,$$

and

$$r_p < r_{p+1}.$$

Since $p + 1 < b_{a+1}$, we have by Equation (2.2) that

$$r_{p+1} < r_{b_a}.$$

By the minimality of p, no row index increase occurs between b_a and p. Hence,

$$r_{b_a} > r_{b_a+1} > \cdots > r_p.$$

Set

$$y := r_{p+1}.$$

Then

$$r_{b_a} > y > r_p.$$

Therefore, there exists an index q with

$$b_a \le q < p$$

such that

$$r_q > y > r_{q+1}.$$

Indeed, the sequence

$$r_{b_a}, r_{b_a+1}, \ldots, r_p$$

is strictly decreasing, starts above y, and ends below y.

Set

$$x := c_q.$$

Since $r_q > r_{q+1}$, Lemma 2.14 implies

$$c_{q+1} = c_q + 1 = x + 1.$$

We claim that

$$\mathrm{rk}(M_\sigma)[y, x] = \mathrm{rk}(M_\sigma)[y+1, x] = \mathrm{rk}(M_\sigma)[y, x+1] = \mathrm{rk}(M_\sigma)[y+1, x+1] \neq 0.$$

Indeed, row $y = r_{p+1}$ contains its unique 1 in column c_{p+1}. Since $q + 1 \le p$, we have

$$x + 1 = c_{q+1} \le c_p < c_{p+1}.$$

Thus, row y contains no 1 in any column with index $\le x+1$. Also, column $x+1 = c_{q+1}$ contains its unique 1 in row r_{q+1}, and

$$r_{q+1} < y.$$

Thus, column $x + 1$ contains no 1 in any row with index $\ge y$. These two emptiness conditions show that

$$\mathrm{rk}(M_\sigma)[y, x] = \mathrm{rk}(M_\sigma)[y+1, x] = \mathrm{rk}(M_\sigma)[y, x+1] = \mathrm{rk}(M_\sigma)[y+1, x+1].$$

Finally, the common value is nonzero because (r_q, c_q) lies in the southwest rectangle

$$\{(u, v) \mid u \ge y + 1,\ v \le x\},$$

since $r_q > y$ and $c_q = x$. Therefore, we have found a contiguous 2×2 block of equal, nonzero entries in the southwest rank table of M_σ, which is a contradiction. □

Lemma 2.16. *Let σ avoid a contiguous 2×2 block of equal, nonzero entries in its southwest rank table. Fix a run beginning at the base point with index b_a. If*

$$b_a \le p < b_{a+1} - 1$$

(or $b_a \le p < t$ when $a = k$), then r_{p+1} is the largest row index strictly less than r_{b_a} that does not occur among

$$r_1, r_2, \ldots, r_p.$$

Proof. Since p lies inside the run beginning at b_a, we have by Lemma 2.15 that

$$r_{b_a} > r_{b_a+1} > \cdots > r_p > r_{p+1}.$$

Suppose for the sake of contradiction that there exists a row index y such that

$$r_{p+1} < y < r_{b_a}$$

and $y \notin \{r_1, \ldots, r_p\}$.

Because $y \notin \{r_1, \ldots, r_p\}$, row y contains no 1 in any column $\le c_p$. By Lemma 2.14,

$$c_{p+1} = c_p + 1.$$

Since the 1 in column c_{p+1} lies in row $r_{p+1} < y$, column $c_p + 1$ contains no 1 in any row $\geq y$. Therefore,

$$\mathrm{rk}(M_\sigma)[y, c_p] = \mathrm{rk}(M_\sigma)[y+1, c_p] = \mathrm{rk}(M_\sigma)[y, c_p+1] = \mathrm{rk}(M_\sigma)[y+1, c_p+1].$$

Additionally, we have that (r_{b_a}, c_{b_a}) lies in the southwest rectangle

$$\{(u, v) \mid u \geq y,\ v \leq c_p\},$$

because $r_{b_a} > y$ and $c_{b_a} \leq c_p$. Hence,

$$\mathrm{rk}(M_\sigma)[y, c_p] = \mathrm{rk}(M_\sigma)[y+1, c_p] = \mathrm{rk}(M_\sigma)[y, c_p+1] = \mathrm{rk}(M_\sigma)[y+1, c_p+1] \neq 0,$$

which is a contiguous 2×2 block of equal, nonzero entries in the southwest rank table. Thus, we reach a contradiction. □

Theorem 2.17. *For the partial permutation matrix of $\sigma \in \mathrm{PM}(m, n)$, there exists some element $\mu \in \mathrm{PP}(m, n)$ such that $\Phi(\mu) = \sigma$ if and only if σ avoids a contiguous 2×2 block of equal, nonzero entries in its southwest rank table.*

Proof. The forward direction is shown in Lemma 2.13.

For the reverse direction, assume that σ avoids a contiguous 2×2 block of equal, nonzero entries in its southwest rank table. Recall the element $\mu \in \mathrm{PP}(m, n)$ constructed from σ in Equation (2.3). Define

$$\sigma' = \Phi(\mu).$$

We will show that $\sigma' = \sigma$.

We only have that σ avoids the 2×2 pattern, hence the structural lemmas apply to σ, not to σ'. Thus, we first use them to determine the arrangement of the 1's in M_σ, and then compare this arrangement with the placements made by the algorithm defining Φ. Additionally, recall that the indices (r_i, c_i) are defined to be the position of 1's in M_σ.

Fix a run in M_σ beginning at the base point indexed by b_a, where $1 \leq a \leq k$. By construction of μ in Equation (2.3), the algorithm in Corollary 2.11 places a base point of $M_{\sigma'}$ at

$$(r_{b_a}, c_{b_a}).$$

This agrees with the corresponding 1 of M_σ.

Now consider the remaining positions in this run in M_σ. By Lemma 2.15, the row indices in the run satisfy

$$r_{b_a} > r_{b_a+1} > \cdots > r_{b_{a+1}-1}$$

if $a < k$, and

$$r_{b_a} > r_{b_a+1} > \cdots > r_t$$

if $a = k$. Applying Lemma 2.14 inductively to these decreasing chains, we get that the columns in each run are consecutive. Namely,

$$c_{b_a+p} = c_{b_a} + p$$

for every p such that $b_a + p$ remains in the run.

We now prove that $\Phi(\mu)$ places the same 1's as σ throughout this run. Suppose that $p \geq 1$ and that $b_a + p$ remains in the run. Assume inductively that, in all earlier columns, the placements of $M_{\sigma'}$ agree with those of M_σ. In particular, the rows already used by $M_{\sigma'}$ in earlier columns are precisely

$$r_1, r_2, \ldots, r_{b_a+p-1}.$$

Since the columns in the run in M_σ are consecutive, the next column under consideration is

$$c_{b_a} + p = c_{b_a+p}.$$

By Lemma 2.16, the row r_{b_a+p} is the largest row index strictly less than r_{b_a} that does not occur among

$$r_1, r_2, \ldots, r_{b_a+p-1}.$$

This is exactly the rule used by the algorithm in Corollary 2.11: starting from the base point (r_{b_a}, c_{b_a}), the algorithm places the next 1 in the next column and in the largest available row

strictly above the base row. Therefore, the 1 placed by $\Phi(\mu)$ in column c_{b_a+p} lies in row r_{b_a+p}. This means $M_{\sigma'}$ has a 1 at

$$(r_{b_a+p}, c_{b_a+p}),$$

which is exactly the corresponding 1 of M_σ.

Thus, by induction within the run, every placement made by $\Phi(\mu)$ in this run agrees with the corresponding 1 of M_σ. Since this argument applies to every run, and since the runs partition the set of 1's of M_σ, the matrices $M_{\sigma'}$ and M_σ have their 1's in exactly the same positions. Therefore,

$$\sigma' = \sigma.$$

Since $\sigma' = \Phi(\mu)$, we conclude that

$$\Phi(\mu) = \sigma.$$

This proves the reverse direction, and hence the theorem. □

Corollary 2.18. If $\sigma \in \mathrm{PM}(m, n)$ avoids a contiguous 2×2 block of equal, nonzero entries in its southwest rank table, then $\overline{\mathcal{O}_\sigma}$ is linear (and thus smooth).

Proof. By Theorem 2.17, we have

$$\overline{\mathcal{O}_\sigma} = \overline{L}_1^\mu$$

for some $\mu \in \mathrm{PP}(m, n)$. Moreover, by Lemma 2.8, we know $\overline{L}_1^\mu$ is linear, so $\overline{\mathcal{O}_\sigma}$ is also linear (and thus smooth). □

2.3. **Smooth Matrix Schubert Varieties Are Linear.** Now that we have a full characterization of the matrix Schubert varieties $\overline{\mathcal{O}_\sigma}$ that are linear, we show that every smooth orbit closure $\overline{\mathcal{O}_\sigma}$ is linear by studying the tangent space dimension at the empty partial matching $\emptyset$.

Lemma 2.19. *The definition of $\overline{\mathcal{O}_\sigma}$ as $B_1 \times B_2$-orbit closures in $\mathrm{Mat}_{n\times m}$ is equivalent to the rank table definition*

$$\overline{\mathcal{O}_\sigma} = \{A \in \mathrm{Mat}_{n\times m} : \mathrm{rk}(A)[r, c] \leq \mathrm{rk}(M_\sigma)[r, c] \text{ for all } 1 \leq r \leq n,\ 1 \leq c \leq m\}, \tag{2.4}$$

where $\mathrm{rk}(A)$ denotes the southwest rank table of the matrix A.

Proof. This is the southwest-convention version of the rectangular matrix Schubert variety theorem. In the northwest convention, the textbook [MS05] defines, for a $k \times \ell$ partial permutation matrix w, the matrix Schubert variety

$$\overline{X_w} = \{Z \in M_{k\times\ell} : \mathrm{rank}(Z_{p\times q}) \leq \mathrm{rank}(w_{p\times q}) \text{ for all } p, q\},$$

where $Z_{p\times q}$ denotes the northwest $p \times q$ submatrix. They prove that $\overline{X_w}$ is the closure of the corresponding $B \times B_+$-orbit of w [MS05, Definition 15.1, Theorem 15.31].

In their convention, the notation B is the lower triangular Borel acting on rows, while B_+ is the upper triangular Borel acting on columns. With these conventions, the natural defining inequalities are northwest rank conditions. Our convention instead uses southwest rank conditions, as both Borels B_1 and B_2 are upper-triangular. Equivalently, our convention is obtained from the convention in [MS05] by reversing the order of the rows. Under this row reversal, northwest submatrices become southwest submatrices, and the northwest rank table in [MS05] becomes the southwest rank table used here. Therefore, the equivalent statement of [MS05, Theorem 15.31] in our convention is

$$\overline{\mathcal{O}_\sigma} = \{A \in \mathrm{Mat}_{n\times m} : \mathrm{rk}(A)[r, c] \leq \mathrm{rk}(M_\sigma)[r, c] \text{ for all } 1 \leq r \leq n,\ 1 \leq c \leq m\},$$

where $\mathrm{rk}(A)$ denotes the southwest rank table. □

These explicit rank conditions cutting out matrix Schubert varieties from $\mathrm{Mat}_{n\times m}$ allow us to compute the tangent space dimension at the empty partial matching for the variety.

Lemma 2.20. *The tangent space to $\overline{\mathcal{O}_\sigma}$ at the empty partial matching $\emptyset \in \mathrm{PM}(m, n)$ is*

$$T_\emptyset(\overline{\mathcal{O}_\sigma}) = \{Z = (z_{i,j}) \in \mathrm{Mat}_{n\times m} \mid z_{r,c} = 0 \text{ whenever } \mathrm{rk}(M_\sigma)[r, c] = 0\}.$$

Consequently,

$$\dim T_\emptyset(\overline{\mathcal{O}_\sigma}) = \#\{(r, c) \mid \mathrm{rk}(M_\sigma)[r, c] > 0\}.$$

Proof. Let $A_{r,c}$ be the southwest submatrix of A with northeast corner (r, c). By Lemma 2.19, the variety $\overline{\mathcal{O}_\sigma}$ is defined by the rank conditions

$$\mathrm{rk}(A_{r,c}) \leq \mathrm{rk}(M_\sigma)[r, c]$$

for all (r, c). Equivalently, if

$$k_{r,c} = \mathrm{rk}(M_\sigma)[r, c],$$

then all $(k_{r,c} + 1) \times (k_{r,c} + 1)$ minors of $A_{r,c}$ vanish.

(1) If $\mathrm{rk}(M_\sigma)[r, c] = 0$. Then every entry in the southwest $r \times c$ rectangle must vanish on O_σ. In particular, the defining equations include the linear equations

$$A_{i,j} = 0 \quad \text{for all } r \leq i \leq n,\ 1 \leq j \leq c.$$

Therefore, for every tangent vector

$$Z = (z_{p,q}) \in T_\emptyset(\overline{\mathcal{O}_\sigma}),$$

we must have $z_{p,q} = 0$ for $r \leq p \leq n,\ 1 \leq q \leq c$. In particular, we have $z_{r,c} = 0$.

(2) If $\mathrm{rk}(M_\sigma)[r, c] = k > 0$. Then each $(k + 1) \times (k + 1)$ minors of $A_{r,c}$ is a homogeneous polynomial of degree $k + 1 \geq 2$ in the entries of A, so its linear part at $t = 0$ vanishes identically. Therefore, these equations impose no linear constraints on the tangent vector $Z \in T_\emptyset(\overline{\mathcal{O}_\sigma})$.

Combining the two cases, the only first-order conditions arise exactly from positions (r, c) with $\mathrm{rk}(M_\sigma)[r, c] = 0$, giving the stated description of the tangent space. The dimension formula follows immediately by counting free coordinates. □

Since the singular locus is closed and stable under the $B_1 \times B_2$-action, and since the closure of every orbit contains the empty partial matching, the orbit closure $\overline{\mathcal{O}_\sigma}$ is smooth if and only if it is smooth at $\emptyset$. Recall that an algebraic variety is smooth at a point x if and only if the local dimension of the variety at x is equal to the dimension of the tangent space at x [CLO07]. Because $\overline{\mathcal{O}_\sigma}$ is the closure of an irreducible B-orbit, it is irreducible. In particular, the local dimension of the variety at any point x is just the dimension of the entire variety. Thus, for matrix Schubert varieties, we can use Lemma 2.20 to compare the tangent space dimension at the empty partial matching to the dimension of $\overline{\mathcal{O}_\sigma}$ to determine smoothness.

Corollary 2.21. If the matrix Schubert variety $\overline{\mathcal{O}_\sigma}$ is smooth, then it is linear.

Proof. Let

$$S_\sigma = \{(r, c) : \mathrm{rk}(M_\sigma)[r, c] > 0\}.$$

By Lemma 2.19, if $\mathrm{rk}(M_\sigma)[r, c] = 0$, then every matrix $A \in \overline{\mathcal{O}_\sigma}$ satisfies

$$\mathrm{rk}(A)[r, c] = 0.$$

Hence, the southwest submatrix of A with northeast corner (r, c) is zero. In particular, the entry $A[r, c]$ is zero. Therefore,

$$\overline{\mathcal{O}_\sigma} \subseteq L_\sigma := \{A \in \mathrm{Mat}_{n\times m} : A[r, c] = 0 \text{ whenever } \mathrm{rk}(M_\sigma)[r, c] = 0\}.$$

The variety L_σ is the coordinate linear subspace whose free coordinates are precisely the positions in S_σ. Thus,

$$\dim L_\sigma = |S_\sigma|.$$

Since $\overline{\mathcal{O}_\sigma}$ is smooth, it is smooth at the empty partial matching $\emptyset$. Moreover, because $\overline{\mathcal{O}_\sigma}$ is the closure of a $B_1 \times B_2$-orbit, it is irreducible. In particular, the local dimension at $\emptyset$ is equal to the dimension of the variety:

$$\dim_\emptyset(\overline{\mathcal{O}_\sigma}) = \dim \overline{\mathcal{O}_\sigma}.$$

By the smoothness criterion at a point,

$$\dim_\emptyset(\overline{\mathcal{O}_\sigma}) = \dim T_\emptyset(\overline{\mathcal{O}_\sigma}).$$

Furthermore, by Lemma 2.20,

$$\dim T_\emptyset(\overline{\mathcal{O}_\sigma}) = |S_\sigma|.$$

Therefore,

$$\dim \overline{\mathcal{O}_\sigma} = |S_\sigma| = \dim L_\sigma.$$

Hence, the orbit closure $\overline{\mathcal{O}_\sigma}$ is a closed irreducible subvariety of the irreducible variety L_σ with the same dimension. Thus,

$$\overline{\mathcal{O}_\sigma} = L_\sigma,$$

and $\overline{\mathcal{O}_\sigma}$ is a coordinate linear subspace. □

We are now ready to prove the first main result of this paper.

Proof of Theorem 1.2. The reverse direction is shown in Corollary 2.18. For the forward direction, suppose that $\overline{\mathcal{O}_\sigma}$ is smooth. Then $\overline{\mathcal{O}_\sigma}$ is linear by Corollary 2.21. Moreover, since $\overline{\mathcal{O}_\sigma}$ is a $B_1 \times B_2$-orbit closure, it is $B_1 \times B_2$-stable. Therefore, by Lemma 2.8, we have

$$\overline{\mathcal{O}_\sigma} = \overline{L}_1^\mu$$

for some $\mu \in \mathrm{PP}(m, n)$. Finally, by Theorem 2.17, it follows that σ avoids a contiguous 2×2 block of equal, nonzero entries in its southwest rank table. □

2.4. **Combinatorial Counting of Smooth Matrix Schubert Varieties.** In this subsection, we count the number of smooth matrix Schubert varieties of a fixed matrix dimension. This section reveals patterns related to the well-known Catalan numbers and highlights the power of the characterization described by Theorem 1.2.

Proposition 2.22. The number of smooth matrix Schubert varieties $\overline{\mathcal{O}_\sigma}$ for $\sigma \in \mathrm{PM}(m, n)$ is $\binom{m+n}{m}$.

Proof. From Theorem 1.2 and Theorem 2.17, we have that the smooth matrix Schubert varieties $\overline{\mathcal{O}_\sigma}$ in $\mathrm{Mat}_{n\times m}$ are in bijection with the elements of $\mathrm{PP}(m, n)$. By Definition 2.6,

$$|\mathrm{PP}(m, n)| = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} = \sum_{k=0}^{m} \binom{m}{m-k}\binom{n}{k},$$

where the binomial coefficient $\binom{n}{k}$ equals 0 if $k > n$. Then, by Vandermonde's identity,

$$|\mathrm{PP}(m, n)| = \sum_{k=0}^{m} \binom{m}{m-k}\binom{n}{k} = \binom{m+n}{m}.$$

□

2.4.1. *Full-rank smooth matrix Schubert varieties.* We can also count the smooth matrix Schubert varieties of a fixed matrix dimension that have full rank, meaning that their representative partial permutation matrix M_σ is full rank. To do this, we first recall the Catalan triangle numbers. The Catalan triangle was introduced by L.W. Shapiro [Sha76] and is depicted in (2.5).

(2.5)

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
1 & 1 & & & & & & & \\
1 & 2 & 2 & & & & & & \\
1 & 3 & 5 & 5 & & & & & \\
1 & 4 & 9 & 14 & 14 & & & & \\
1 & 5 & 14 & 28 & 42 & 42 & & & \\
1 & 6 & 20 & 48 & 90 & 132 & 132 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

For $1 \le q \le p$, let $C(p, q)$ denote the number in row $p + 1$ and column $q + 1$ in the Catalan triangle. The following definition records the recursion satisfied by these numbers.

Definition 2.23. For $p \geq 1$ and $1 \leq q \leq p$, the Catalan triangle number $C(p, q)$ is recursively defined by

$$C(p,q) = \begin{cases} p, & q = 1, \\ C(p, q-1) + C(p-1, q), & 1 < q < p, \\ C(p, p-1), & q = p > 1. \end{cases}$$

We shall also use the standard closed form

$$C(p,q) = \frac{p-q+1}{p+1}\binom{p+q}{q}. \tag{2.6}$$

Theorem 2.24. *For positive integers $n \geq m$, the number of smooth full-rank orbit closures $\overline{\mathcal{O}_\sigma}$ for $\sigma \in \mathrm{PM}(m, n)$ is*

$$C(n,m) = \frac{n-m+1}{n+1}\binom{n+m}{m}.$$

Proof. Recall the convention that an element of $\mathrm{PM}(m, n)$ is represented by a partial matching matrix of $n \times m$. Let $S(n, m)$ denote the set of full-rank $n \times m$ partial matching matrices whose corresponding matrix Schubert varieties are smooth. Additionally, we will refer to a contiguous 2×2 block of equal, nonzero entries as a forbidden block. Note that by Theorem 1.2, the definition of $S(n, m)$ is equivalent to the set of full-rank $n \times m$ partial matching matrices not containing any forbidden blocks.

Furthermore, call a row or column in a partial matching matrix empty when the row or column consists of only zeros and nonempty otherwise. We will show that $|S(n, m)|$ satisfies the recursion in Definition 2.23.

Case 1: $m = 1$. The southwest rank table of any full-rank $n \times 1$ partial matching has only one column, so it cannot contain a forbidden block. There are n possible positions for the unique 1, so therefore

$$|S(n, 1)| = n = C(n, 1).$$

Case 2: $1 < m < n$. We partition $S(n, m)$ into two disjoint subsets. Let $S_1(n, m)$ be the subset consisting of partial matching matrices with a nonempty row n, and let $S_2(n, m)$ be the subset consisting of partial matching matrices with an empty row n. Therefore,

$$S(n, m) = S_1(n, m) \sqcup S_2(n, m).$$

We first show that $S_1(n, m)$ is in bijection with $S(n, m-1)$. Consider

$$f : S(n, m-1) \longrightarrow S_1(n, m)$$

where for $M_\sigma \in S(n, m-1)$, the image $f(M_\sigma)$ is obtained by adding a new column to the right and placing a 1 in column m of the lowest empty row of M_σ. Since M_σ has rank $m-1$ on n rows and $m-1 < n$, exactly $n-(m-1) \geq 1$ rows of M_σ are empty, so the lowest empty row is well defined. Placing a 1 in a previously empty row increases the rank by exactly 1, so $f(M_\sigma)$ has full rank. Moreover, the image $f(M_\sigma)$ has a nonempty row n since the row was already nonempty in M_σ or the new 1 is placed in row n, which makes it nonempty.

We now show that $f(M_\sigma)$ still avoids forbidden blocks. The southwest rank table entries from columns 1 to $m-1$ are the same for M_σ and $f(M_\sigma)$, so any forbidden blocks in $f(M_\sigma)$ must contain column m.

Let the newly added 1 lie in row r. Note that for all $r < s < n$, we have

$$\mathrm{rk}(f(M_\sigma))[s, m-1] \geq \mathrm{rk}(f(M_\sigma))[s+1, m-1].$$

This is because every 1 counted by $\mathrm{rk}(f(M_\sigma))[s+1, m-1]$ lies in a row of index at least $s+1$, and therefore also has row index at least s, and so is counted by $\mathrm{rk}(f(M_\sigma))[s, m-1]$ as well. Moreover, this inequality is strict. Since r was the lowest empty row prior to adding the new 1 and $s > r$, row s was already nonempty in M_σ, with its unique 1 lying in some column less than m. That 1 lies in row s, so it contributes to $\mathrm{rk}(f(M_\sigma))[s, m-1]$ but not to $\mathrm{rk}(f(M_\sigma))[s+1, m-1]$. Therefore,

$$\mathrm{rk}(f(M_\sigma))[s, m-1] > \mathrm{rk}(f(M_\sigma))[s+1, m-1].$$

So, no forbidden block containing column m can consist of rows greater than r.

Similarly, for all $1 \leq s \leq r$, we have

$$\mathrm{rk}(f(M_\sigma))[s, m-1] \leq \mathrm{rk}(f(M_\sigma))[s, m].$$

This is because every 1 counted by $\mathrm{rk}(f(M_\sigma))[s, m-1]$ lies in a column less than m, hence also in a column less than or equal to m, and so is counted by $\mathrm{rk}(f(M_\sigma))[s, m]$ as well. Moreover, this inequality is strict. The newly added 1 lies in row r and column m, and since $s \leq r$, this 1 lies in a row greater than or equal to s. So, it contributes to $\mathrm{rk}(f(M_\sigma))[s, m]$. However, since this 1 is in column m, it does not contribute to $\mathrm{rk}(f(M_\sigma))[s, m-1]$. Therefore,

$$\mathrm{rk}(f(M_\sigma))[s, m-1] < \mathrm{rk}(f(M_\sigma))[s, m].$$

So, no forbidden block containing column m can consist of rows less than or equal to r.

Therefore, there are no forbidden 2×2 blocks, which implies that $f(M_\sigma)$ lies in $S_1(n, m)$ and f is well defined.

Now consider

$$g : S_1(n, m) \longrightarrow S(n, m-1)$$

where for $M_\sigma \in S_1(n, m)$, the image $g(M_\sigma)$ is obtained by deleting column m. The resulting partial matching matrix has full rank $m-1$ since every column still has exactly one 1. Moreover, deleting column m removes column m of the southwest rank table and leaves all remaining entries unchanged. Thus, no forbidden block is created, which implies $g(M_\sigma)$ lies in $S(n, m-1)$ and g is well defined.

It is immediate that $g \circ f$ is the identity on $S(n, m-1)$, as adding and then removing a column does nothing. We now show that $f \circ g$ is the identity on $S_1(n, m)$. Let $M_\sigma \in S_1(n, m)$, and suppose that the 1 in column m of M_σ lies in row r. After applying g, row r becomes empty. It is sufficient to show that row r is the lowest empty row in $g(M_\sigma)$, so that f places the 1 back in row r.

If $r = n$, this is clear. Otherwise, in M_σ, row n is nonempty and the 1 is not in column m, so row n remains nonempty in $g(M_\sigma)$. Assume for the sake of contradiction that there is an empty row $s > r$ in $g(M_\sigma)$. Since row n is nonempty in $g(M_\sigma)$, we have $s < n$. Note that a 1 contributes to $\mathrm{rk}(g(M_\sigma))[s, m-1]$ if it is in row s or below and contributes to $\mathrm{rk}(g(M_\sigma))[s+1, m-1]$ if it is in row $s+1$ or below. Since row s is empty, we get

$$\mathrm{rk}(g(M_\sigma))[s, m-1] = \mathrm{rk}(g(M_\sigma))[s+1, m-1]$$

In M_σ, the deleted 1 in row r does not contribute to the southwest ranks in rows s and $s+1$, since $s, s+1 > r$. Therefore, the four entries in rows s and $s+1$ of columns $m-1$ and m of M_σ's southwest rank table form a forbidden block, a contradiction with $M_\sigma \in S_1(n, m)$. Hence, there is no empty row below r in $g(M_\sigma)$ and f places the 1 back in row r. So, the functions f and g are inverses, which implies there is a bijection between $S_1(n, m)$ and $S(n, m-1)$. So, we get

$$|S_1(n, m)| = |S(n, m-1)|.$$

Next, we show that $S_2(n, m)$ is in bijection with $S(n-1, m)$. Consider

$$f : S_2(n, m) \longrightarrow S(n-1, m)$$

where for $M_\sigma \in S_2(n, m)$, the image $f(M_\sigma)$ is obtained by removing row n. Note that this preserves full rank since the removed row is empty as $M_\sigma \in S_2(n, m)$. Additionally, the southwest rank table of $f(M_\sigma)$ is obtained from the southwest rank table of M_σ by deleting the row n, so no forbidden block is created. Thus, function f is well defined.

Now consider

$$g : S(n-1, m) \longrightarrow S_2(n, m)$$

where for $M_\sigma \in S(n-1, m)$, the image $g(M_\sigma)$ is obtained by adding an empty row n, which preserves full rank since $m \leq n-1$. The southwest rank table of $g(M_\sigma)$ is obtained from the southwest rank table of M_σ by adding a row of zeros at the bottom. Since the forbidden blocks must have nonzero entries, no forbidden block is created. Thus, function g is well defined.

It is clear that these two functions are inverse to each other and are thus bijections, so

$$|S_2(n,m)| = |S(n-1,m)|.$$

Therefore, for $1 < m < n$,

$$|S(n,m)| = |S_1(n,m)| + |S_2(n,m)| = |S(n,m-1)| + |S(n-1,m)|.$$

Case 3: $m = n > 1$. In this case, we show that $S(n,n)$ is in bijection with $S(n,n-1)$. Define

$$f : S(n,n) \longrightarrow S(n,n-1)$$

where for $M_\sigma \in S(n,n)$, the image $f(M_\sigma)$ is obtained by removing column n. Note that M_σ has rank n and $f(M_\sigma)$ has rank $n-1$. Thus, the resulting $n \times (n-1)$ matrix still has full rank. Additionally, the southwest rank table of $f(M_\sigma)$ is obtained from the southwest rank table of M_σ by deleting column n, so no forbidden 2×2 block is created. Thus, function f is well defined.

Conversely, define

$$g : S(n,n-1) \longrightarrow S(n,n)$$

where for $M_\sigma \in S(n,n-1)$, the image $g(M_\sigma)$ is obtained by adding a new column on the right and placing a 1 in the unique empty row of M_σ. We know there is exactly one empty row in M_σ since the rank is $n-1$ but there are n rows. Additionally, full rank is preserved and columns 1 through $n-1$ of the southwest rank table remain unchanged. So, any newly created forbidden block must contain part of the column n. However, because the resulting partial matching has one 1 in every row, adjacent entries in column n of $g(M_\sigma)$'s southwest rank table differ. Therefore, no forbidden block can involve the column n. Thus, function g is well defined.

These two operations are clearly inverse to each other and are thus bijections, so

$$|S(n,n)| = |S(n,n-1)|.$$

Since $|S(n,m)|$ satisfies the same initial condition, recursion, and boundary condition as the Catalan triangle numbers $C(n,m)$, we get

$$|S(n,m)| = C(n,m).$$

□

Corollary 2.25. For positive integers m and n, the number of smooth full-rank orbit closures $\overline{\mathcal{O}_\sigma}$ for $\sigma \in \mathrm{PM}(m,n)$ is

$$C(\max(m,n), \min(m,n)).$$

Proof. This corollary generalizes Theorem 2.24 by removing the condition that $n \geq m$.

If $n \geq m$, then Theorem 2.24 applies directly and gives

$$C(n,m) = C(\max(m,n), \min(m,n)).$$

Now suppose $m > n$. Reflecting an $n \times m$ partial matching across its antidiagonal gives an $m \times n$ partial matching, or an element of $\mathrm{PM}(n,m)$. This operation preserves rank, so the image is full rank if and only if the original partial matching is full rank.

Moreover, antidiagonal reflection preserves the forbidden 2×2 pattern in Theorem 1.2. Indeed, the southwest rank table of the reflected partial matching is obtained by reflecting the original southwest rank table across its antidiagonal. Therefore, the reflected partial matching has a contiguous 2×2 block of equal, nonzero entries in its southwest rank table if and only if the original partial matching does. Thus, antidiagonal reflection gives a bijection between smooth full-rank elements of $\mathrm{PM}(m,n)$ and smooth full-rank elements of $\mathrm{PM}(n,m)$.

Since $m > n$, Theorem 2.24 applies to $\mathrm{PM}(n,m)$ and gives $C(m,n)$ smooth full-rank orbit closures. Therefore, the number of smooth full-rank orbit closures for $\sigma \in \mathrm{PM}(m,n)$ is

$$C(m,n) = C(\max(m,n), \min(m,n)).$$

□

Corollary 2.26. For any positive integer n, the number of smooth full-rank orbit closures $\overline{\mathcal{O}_\sigma}$ for $\sigma \in \mathrm{PM}(n,n)$ is the n-th Catalan number.

2.4.2. *Fixed-rank smooth matrix Schubert varieties.* For nonnegative integers m, n, let

$$\mathcal{H}(m,n) = \{h = (h_1, \ldots, h_m) \mid 0 \le h_1 \le \cdots \le h_m \le n\}.$$

We interpret an element of $\mathcal{H}(m,n)$ as the height vector of a staircase in an $n \times m$ rectangle. More explicitly, if

$$\mu = (I, J) \in \mathrm{PP}(m,n), \qquad I = \{i_1 < \cdots < i_t\}, \qquad J = \{j_1 < \cdots < j_t\},$$

then define

$$h_c(\mu) = \max\big(\{0\} \cup \{j_a \mid i_a \le c\}\big), \qquad 1 \le c \le m.$$

This gives a bijection between $\mathrm{PP}(m,n)$ and $\mathcal{H}(m,n)$. The inverse sends $h = (h_1, \ldots, h_m)$ to the element $\mu(h) = (I(h), J(h)) \in \mathrm{PP}(m,n)$ defined by setting $h_0 = 0$ and

$$I(h) = \{c \in \{1, \ldots, m\} \mid h_c > h_{c-1}\}, \qquad J(h) = \{h_c \mid h_c > h_{c-1}\}.$$

Under this convention, the bottom row of the staircase is occupied if and only if $h_m = n$, and the leftmost column of the staircase is occupied if and only if $h_1 > 0$.

For $h \in \mathcal{H}(m,n)$, define

$$\rho(h) = \mathrm{rk}\big(M_{\Phi(\mu(h))}\big).$$

Let

$$\mathcal{H}(m,n,k) := \{h \in \mathcal{H}(m,n) \mid \rho(h) = k\},$$

and also define

$$\Delta(h) = \max(0,\ 1 - h_1,\ 2 - h_2,\ \ldots,\ m - h_m).$$

Lemma 2.27. *For every $h = (h_1, \ldots, h_m) \in \mathcal{H}(m,n)$, we have*

$$\rho(h) = m - \Delta(h).$$

Proof. Let e_c denote the number of empty columns in $M_{\Phi(\mu(h))}$ among the columns indexed $1, \ldots, c$. We show by induction on c that

$$e_c = \max(0, 1 - h_1, \ldots, c - h_c)$$

for all $1 \le c \le m$.

Consider the base case $c = 1$. If $h_1 = 0$, the first column of $M_{\Phi(\mu(h))}$ is empty, or $e_1 = 1$. Otherwise, if $h_1 > 0$, then the first column has a 1 and $e_1 = 0$. Both cases satisfy

$$e_1 = \max(0, 1 - h_1).$$

Now, assume the inductive hypothesis holds for some fixed $1 \le c < m$. We first show that

$$e_{c+1} = \max(e_c, c + 1 - h_{c+1}\}.$$

Recall the algorithm for Φ from Lemma 2.10. Suppose that we have already processed columns $x = 1, \ldots, c$. In column $c+1$, exactly h_{c+1} rows are available, and $c - e_c$ of these rows have already been used. Therefore, the size of the stack S before processing column $c+1$ is

$$h_{c+1} - (c - e_c).$$

We have two cases.

Case 1: $h_{c+1} - (c - e_c) > 0$. This means that the stack is nonempty. Thus, column $c+1$ receives a 1, and the number of empty columns does not change:

$$e_{c+1} = e_c.$$

Moreover, because all quantities are integers,

$$\begin{aligned} h_{c+1} - (c - e_c) &> 0 \\ e_c &> c - h_{c+1} \\ e_c &\ge c + 1 - h_{c+1}. \end{aligned}$$

Thus, in this case, we have

$$e_{c+1} = \max(e_c, c + 1 - h_{c+1}).$$

Case 2: $h_{c+1} - (c - e_c) = 0$. This means that the stack is empty, and thus, column $c+1$ is also empty. Hence,

$$e_{c+1} = e_c + 1.$$

Moreover,

$$\begin{aligned} h_{c+1} - (c - e_c) &= 0 \\ e_c &= c - h_{c+1} \\ e_c + 1 &= c + 1 - h_{c+1}. \end{aligned}$$

Thus, we again have

$$e_{c+1} = e_c + 1 = \max(e_c, e_c + 1) = \max(e_c, c + 1 - h_{c+1}).$$

Then, by the inductive assumption,

$$\begin{aligned} e_{c+1} = \max(e_c, c + 1 - h_{c+1}) &= \max(\max(0, 1 - h_1, \ldots, c - h_c), c + 1 - h_{c+1}) \\ &= \max(0, 1 - h_1, \ldots, c + 1 - h_{c+1}), \end{aligned}$$

which completes the induction.

At $c = m$, we have

$$e_m = \max(0, 1 - h_1, \ldots, m - h_m) = \Delta(h).$$

Finally, by definition of e_m,

$$\rho(h) = m - e_m = m - \Delta(h).$$

□

Lemma 2.28. *Let m, n be positive integers and let*

$$0 \leq k \leq \min(m, n - 1).$$

Then there is a bijection

$$\Psi_{m,n,k} : \mathcal{H}(m, n, k) \longrightarrow \mathcal{H}(m + 1, n - 1, k).$$

Proof. We construct $\Psi_{m,n,k}$ by induction on k. It is harmless in the induction to allow height vectors with 0 rows or 0 columns; for example, the set $\mathcal{H}(m, 0, 0)$ consists only of the zero vector, and $\mathcal{H}(0, n, 0)$ consists only of the empty vector.

If $k = 0$, then $\mathcal{H}(m, n, 0)$ consists only of the zero height vector $\underbrace{(0, \ldots, 0)}_{m}$ and we define

$$\Psi_{m,n,0} \underbrace{(0, \ldots, 0)}_{m} = \underbrace{(0, \ldots, 0)}_{m+1}.$$

Now assume $k > 0$, and assume by induction that all lower-rank maps $\Psi_{a,b,k-1}$ in the required range have already been constructed and are bijections.

Let

$$h = (h_1, \ldots, h_m) \in \mathcal{H}(m, n, k).$$

We distinguish two cases.

Case 1: $h_m < n$**.** In this case, the bottom row of the staircase is empty. Define

$$\Psi_{m,n,k}(h) = (0, h_1, \ldots, h_m).$$

Since $h_m < n$, all entries of $(0, h_1, \ldots, h_m)$ are at most $n - 1$, so this is an element of $\mathcal{H}(m + 1, n - 1)$. Moreover,

$$\begin{aligned} \Delta(0, h_1, \ldots, h_m) &= \max\{0,\ 1,\ 2 - h_1,\ 3 - h_2, \ldots,\ m + 1 - h_m\} \\ &= 1 + \max\{0,\ 1 - h_1,\ 2 - h_2, \ldots,\ m - h_m\} \\ &= \Delta(h) + 1. \end{aligned}$$

Since $\rho(h) = k$, Lemma 2.27 gives $\Delta(h) = m - k$. Therefore

$$\Delta(\Psi_{m,n,k}(h)) = m + 1 - k,$$

and hence

$$\rho(\Psi_{m,n,k}(h)) = (m + 1) - (m + 1 - k) = k.$$

Thus $\Psi_{m,n,k}(h) \in \mathcal{H}(m+1, n-1, k)$.

Case 2: $h_m = n$**.** In this case, the bottom row of the staircase is occupied. Write

$$h^- = (h_1, \ldots, h_{m-1}).$$

We first show that $h^- \in \mathcal{H}(m-1, n, k-1)$. Since $h_m = n$, we have

$$\Delta(h) = \max\{\Delta(h^-),\, m-n\}.$$

On the other hand, we have $\rho(h) = k$, so $\Delta(h) = m - k$. Because $k \leq n-1$, we have

$$m - k \geq m - n + 1 > m - n.$$

Therefore the final term $m - n$ is not responsible for the maximum, and hence

$$\Delta(h^-) = \Delta(h).$$

It follows that

$$\rho(h^-) = (m-1) - \Delta(h^-) = (m-1) - \Delta(h) = k - 1.$$

Thus

$$h^- \in \mathcal{H}(m-1, n, k-1).$$

Now apply the lower-rank bijection twice:

$$u = \Psi_{m-1,n,k-1}(h^-) \in \mathcal{H}(m, n-1, k-1),$$

and

$$v = \Psi_{m,n-1,k-1}(u) \in \mathcal{H}(m+1, n-2, k-1).$$

Finally define

$$\Psi_{m,n,k}(h) = v + \underbrace{(1, \ldots, 1)}_{m+1}.$$

This height vector has $m+1$ entries and all entries are between 1 and $n-1$, so it lies in $\mathcal{H}(m+1, n-1)$.

We check the rank. Since $v \in \mathcal{H}(m+1, n-2, k-1)$,

$$\Delta(v) = (m+1) - (k-1) = m + 2 - k.$$

Because $k \leq m$, we have $\Delta(v) \geq 2$. Hence adding 1 to every entry decreases Δ by 1:

$$\Delta(v + (1, \ldots, 1)) = \Delta(v) - 1 = m + 1 - k.$$

Therefore

$$\rho(v + (1, \ldots, 1)) = (m+1) - (m+1-k) = k.$$

So $\Psi_{m,n,k}(h) \in \mathcal{H}(m+1, n-1, k)$.

It remains to prove that $\Psi_{m,n,k}$ is a bijection. We construct its inverse

$$\Theta_{m,n,k} : \mathcal{H}(m+1, n-1, k) \longrightarrow \mathcal{H}(m, n, k).$$

Let

$$g = (g_1, \ldots, g_{m+1}) \in \mathcal{H}(m+1, n-1, k).$$

If $g_1 = 0$, define

$$\Theta_{m,n,k}(g) = (g_2, \ldots, g_{m+1}).$$

The same computation as in Case 1, read backwards, shows that this lies in $\mathcal{H}(m, n, k)$.

If $g_1 > 0$, define

$$v = g - (1, \ldots, 1).$$

Then

$$v \in \mathcal{H}(m+1, n-2, k-1).$$

Indeed, since $g_1 > 0$ and g is weakly increasing, all entries of v are nonnegative and at most $n-2$. Also

$$\Delta(g) = (m+1) - k \geq 1,$$

because $k \leq m$. Therefore subtracting 1 from every entry increases Δ by 1:

$$\Delta(v) = \Delta(g) + 1 = m + 2 - k.$$

Thus

$$\rho(v) = (m+1) - (m+2-k) = k-1.$$

Now apply the lower-rank inverses in the reverse order:

$$u = \Theta_{m,n-1,k-1}(v) \in \mathcal{H}(m, n-1, k-1),$$

and

$$h^- = \Theta_{m-1,n,k-1}(u) \in \mathcal{H}(m-1, n, k-1).$$

Finally define

$$\Theta_{m,n,k}(g) = (h^-, n).$$

We now check that $\Theta_{m,n,k}$ and $\Psi_{m,n,k}$ are inverse maps. If $h_m < n$, then

$$\Psi_{m,n,k}(h) = (0, h_1, \ldots, h_m),$$

and $\Theta_{m,n,k}$ deletes the first zero entry, giving back h.

If $h_m = n$, then the construction of $\Psi_{m,n,k}$ first deletes the final entry n, applies two lower-rank maps, and then adds $(1, \ldots, 1)$. The inverse sees that the first entry is positive, subtracts $(1, \ldots, 1)$, applies the two lower-rank inverses, and appends n. By the induction hypothesis, this recovers h.

Conversely, if $g_1 = 0$, then $\Theta_{m,n,k}$ deletes the first entry. Since the resulting height vector has last entry at most $n-1 < n$, applying $\Psi_{m,n,k}$ uses Case 1 and puts the initial zero back. If $g_1 > 0$, then $\Theta_{m,n,k}$ subtracts $(1, \ldots, 1)$, applies the two lower-rank inverses, and appends n. Applying $\Psi_{m,n,k}$ then uses Case 2 and reverses these steps, again by the induction hypothesis. Hence

$$\Theta_{m,n,k} \circ \Psi_{m,n,k} = \mathrm{id}, \qquad \Psi_{m,n,k} \circ \Theta_{m,n,k} = \mathrm{id}\,.$$

Therefore $\Psi_{m,n,k}$ is a bijection. □

We now reach the main result of this section, which is an explicit formula for the number of smooth matrix Schubert varieties of a fixed dimension and rank.

Theorem 2.29. *For positive integers m, n and $0 \leq k \leq \min(n, m)$, the number of partial matchings $\sigma \in \mathrm{PM}(m, n)$ such that $\mathrm{rk}(M_\sigma) = k$ and $\overline{\mathcal{O}_\sigma}$ is smooth is*

$$C(n+m-k, k) = \frac{n+m-2k+1}{n+m-k+1}\binom{n+m}{k}.$$

Proof. When $k = 0$, the only possible partial matching $\sigma \in \mathrm{PM}(m, n)$ is the empty partial matching. Since its southwest rank table contains only zeros, its corresponding $\overline{\mathcal{O}_\sigma}$ is smooth by Theorem 1.2. Indeed, we have

$$\frac{n+m-2k+1}{n+m-k+1}\binom{n+m}{k} = \frac{n+m+1}{n+m+1}\binom{n+m}{0} = 1.$$

When $k > 0$, define

$$f(m, n, k) = \#\left\{\sigma \in \mathrm{PM}(m, n) \;\middle|\; \mathrm{rk}(M_\sigma) = k \text{ and } \overline{\mathcal{O}_\sigma} \text{ is smooth}\right\}.$$

Repeatedly applying Lemma 2.28, we have

$$f(m, n, k) = f(m-1, n+1, k) = f(m-2, n+2, k) = \cdots = f(m-(m-k), n+(m-k), k).$$

Then,

$$f(m-(m-k), n+(m-k), k) = f(k, n+m-k, k)$$

is the full rank case. Because $n \geq k$ and $m \geq k$, we have

$$n + m \geq 2k$$
$$n + m - k \geq k.$$

Hence, Theorem 2.24 gives that

$$f(k, n+m-k, k) = C(n+m-k, k).$$

Finally, applying the explicit form for Catalan Triangle numbers from Equation (2.6) gives

$$f(k, n+m-k, k) = C(n+m-k, k) = \frac{n+m-2k+1}{n+m-k+1}\binom{n+m}{k}.$$

□

3. Smoothness Characterization for Borel Orbit Closures in $\mathrm{FL}(V) \times V$

In this section, we prove a pattern-avoidance characterization for smooth B-orbit closures in $\mathrm{FL}(V) \times V$. We first compute the dimension of the orbit closure indexed by an antichain (w, α). Then, we compute the dimension of its tangent space at the standard base point $(E_\bullet, 0)$, where $E_\bullet$ denotes the standard flag. Comparing the two dimensions gives the smoothness characterization in Theorem 1.6.

3.1. Preliminaries on Borel Orbits in $\mathrm{FL}(V) \times V$.

We begin with the necessary orbit parametrization and related combinatorial notions, following [Tra09].

Let $V = \mathbb{C}^n$, let $\mathrm{FL}(V)$ be the complete flag variety of $\mathrm{GL}(V)$, and let $B \subseteq \mathrm{GL}(V)$ be the Borel subgroup of invertible upper-triangular matrices. Set

$$X = \mathrm{FL}(V) \times V.$$

The group B acts diagonally on X by

$$b \cdot (F_\bullet, v) = (bF_\bullet, bv).$$

The objects studied in this section are the closures of these B-orbits.

We first describe the parametrization of the B-orbits in X.

Definition 3.1. (1) A *permutation of n with an order ideal* is a pair (w, β) where $w \in S_n$ and $\beta \subseteq [n]$ such that if $i \in \beta$ and $j \in \beta^c$, then either $i < j$ or $w(i) < w(j)$. We denote the set of all pairs (w, β) by $\mathrm{RB}'(n)$.

(2) A *permutation of n with an antichain* is a pair (w, α) where $w \in S_n$ and $\alpha \subseteq [n]$ such that if $i, j \in \alpha$ with $i < j$, then $w(i) > w(j)$. We denote the set of all pairs (w, α) by $\mathrm{RB}(n)$.

Remark 3.2. For a fixed $w \in S_n$, let $[n]_w$ be the poset defined by

$$i <_w j \quad \Longleftrightarrow \quad i < j \text{ and } w(i) < w(j).$$

As explained in [Tra09, Lemma 2], the standard correspondence between antichains and order ideals gives a natural bijection

$$\mathrm{RB}(n) \xrightarrow{\sim} \mathrm{RB}'(n), \qquad (w, \alpha) \longmapsto (w, \downarrow_w \alpha),$$

where $\downarrow_w \alpha$ is the order ideal generated by α in $[n]_w$. We write $\beta = \downarrow_w \alpha$.

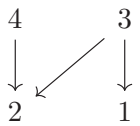
We visualize elements of $\mathrm{RB}'(n)$ and $\mathrm{RB}(n)$ as permutations whose entries are colored either blue or red. We use the convention complementary to that of [Tra09]: for $(w, \beta) \in \mathrm{RB}'(n)$, we color $w(i)$ red if $i \in \beta$ and blue otherwise. Similarly, for $(w, \alpha) \in \mathrm{RB}(n)$, we color $w(i)$ red if $i \in \alpha$ and blue otherwise. Thus, the defining condition of $\mathrm{RB}(n)$ says precisely that the red entries form a decreasing subsequence.

To visualize the poset $[n]_w$, we can use a Hasse diagram. The Hasse diagram for $[n]_w$ is a directed graph, where the vertices are $[n]$ and there is a directed edge from i to j if $i >_w j$ and there is no other vertex $k \in [n]$ satisfying $i >_w k >_w j$.

Example 3.3. Let $w = (3, 1, 4, 2)$. Then, the partial order on $\{1, 2, 3, 4\}$ is given by

$$2 <_w 4, 2 <_w 3, 1 <_w 3.$$

The Hasse diagram is given by



The set of order ideals is

$$\beta = \{\emptyset, \{4,2\}, \{3,2,1\}, \{3,4,2,1\}, \{1\}, \{1,2\}, \{2\}, \{1,4,2\}\}.$$

The set of antichains is

$$\alpha = \{\emptyset, \{4\}, \{3\}, \{3,4\}, \{1\}, \{1,2\}, \{2\}, \{1,4\}\}.$$

They are listed in order with respect to the bijection.

Lemma 3.4. [Tra09] *The B-orbits in $X = \mathrm{FL}(V) \times V$ are parametrized by $\mathrm{RB}'(n)$, or equivalently by $\mathrm{RB}(n)$.*

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $V = \mathbb{C}^n$, and set

$$E_i := \mathrm{span}(e_1, \ldots, e_i), \qquad 0 \le i \le n.$$

Thus $E_\bullet$ is the standard flag, with $E_0 = \{0\}$. For any $F_\bullet \in \mathrm{FL}(V)$, we denote its i-dimensional subspace by F_i. For $w \in S_n$, write $E_\bullet^w := w \cdot E_\bullet$; then

$$E_i^w = wE_i = \mathrm{span}(e_{w(1)}, \ldots, e_{w(i)}).$$

The B-orbit corresponding to $(w, \alpha) \in \mathrm{RB}(n)$, denoted by $\mathcal{O}_{(w,\alpha)}$, has representative

$$(E_\bullet^w, v_\alpha), \qquad v_\alpha = \sum_{a \in \alpha} e_{w(a)}.$$

Thus, we have a stratification

$$X = \mathrm{FL}(V) \times V = \bigsqcup_{(w,\alpha) \in \mathrm{RB}(n)} \mathcal{O}_{(w,\alpha)}.$$

The closures of the B-orbits $\mathcal{O}_{(w,\alpha)}$ are denoted by $\overline{\mathcal{O}_{(w,\alpha)}}$. Our goal is to characterize the smoothness of these orbit closures $\overline{\mathcal{O}_{(w,\alpha)}}$. Note that $\overline{\mathcal{O}_{(w,\alpha)}}$ is a union of B-orbits. This gives a partial order on $\mathrm{RB}(n)$, defined by

$$(w', \alpha') \le (w, \alpha) \quad \text{if and only if} \quad \mathcal{O}_{(w',\alpha')} \subseteq \overline{\mathcal{O}_{(w,\alpha)}}.$$

We refer to this as the Bruhat order on $\mathrm{RB}(n)$.

3.2. **Dimension of the Borel Orbits.** In this subsection, we compute the dimension of each B-orbit $\mathcal{O}_{(w,\alpha)}$. We write

$$\ell(w, \alpha) := \dim \mathcal{O}_{(w,\alpha)}.$$

Lemma 3.5. *For each $(w, \alpha) \in \mathrm{RB}(n)$,*

$$\dim \mathcal{O}_{(w,\alpha)} = \dim \overline{\mathcal{O}_{(w,\alpha)}} = \ell(w, \alpha) = \ell(w) + |\downarrow_w \alpha|.$$

Proof. Set $\beta = \downarrow_w \alpha$. Since

$$H_w := \mathrm{Stab}_B(E_\bullet^w) = B \cap wBw^{-1},$$

consider the projection

$$\pi : \mathrm{FL}(V) \times V \longrightarrow \mathrm{FL}(V).$$

Its restriction to $\mathcal{O}_{(w,\alpha)}$ maps onto $B \cdot E_\bullet^w$, and the fiber over $E_\bullet^w$ is $H_w \cdot v_\alpha$. Since all fibers are translates of this fiber, we have

$$\dim \mathcal{O}_{(w,\alpha)} = \dim(B \cdot E_\bullet^w) + \dim(H_w \cdot v_\alpha) = \ell(w) + \dim(H_w \cdot v_\alpha).$$

It remains to compute the second term. For $h \in H_w$, the entry $h_{w(i),w(j)}$ can be nonzero only if $i \le j$ and $w(i) \le w(j)$. Hence

$$hv_\alpha \in \mathrm{span}\{e_{w(i)} \mid i \in \beta\}.$$

Moreover, for $a \in \alpha$, the coefficient of $e_{w(a)}$ in hv_α is $h_{w(a),w(a)} \neq 0$: a contribution from $e_{w(a')}$ would require either $a = a'$ or $a <_w a'$, and the latter is impossible because α is an antichain. Therefore,

$$H_w \cdot v_\alpha \subseteq S := \left\{ \sum_{i \in \beta} c_i e_{w(i)} \;\middle|\; c_a \neq 0 \text{ for every } a \in \alpha \right\}.$$

Conversely, take $u = \sum_{i\in\beta} c_i e_{w(i)} \in S$. For each $i \in \beta\setminus\alpha$, choose $a(i) \in \alpha$ such that $i <_w a(i)$. Define h by

$$h_{w(a),w(a)} = c_a \quad (a \in \alpha), \qquad h_{w(i),w(a(i))} = c_i \quad (i \in \beta \setminus \alpha),$$

put 1 in the remaining diagonal entries, and put 0 in all remaining off-diagonal entries. The prescribed off-diagonal entries are allowed in both B and wBw^{-1}, while all diagonal entries are nonzero. Thus $h \in H_w$, and by construction $hv_\alpha = u$. Hence $H_w \cdot v_\alpha = S$.

The set S is a nonempty Zariski-open subset of the $|\beta|$-dimensional space $\operatorname{span}\{e_{w(i)} \mid i \in \beta\}$. Consequently,

$$\dim \mathcal{O}_{(w,\alpha)} = \ell(w) + |\beta|.$$

Finally, $\mathcal{O}_{(w,\alpha)}$ is dense in its closure $\overline{\mathcal{O}_{(w,\alpha)}}$, so $\dim \overline{\mathcal{O}_{(w,\alpha)}} = \dim \mathcal{O}_{(w,\alpha)}$. □

3.3. **The Tangent Space at the Base Point.** We next compute the tangent space to $\overline{\mathcal{O}_{(w,\alpha)}}$ at the base point $x_0 = (E_\bullet, 0)$. The following lemma reduces global smoothness to smoothness at this point.

Lemma 3.6. *Every B-orbit closure in $\mathrm{FL}(V) \times V$ contains x_0. Consequently, $\overline{\mathcal{O}_{(w,\alpha)}}$ is smooth if and only if it is smooth at x_0.*

Proof. For any $y = (F_\bullet, v)$, the scalar matrices $tI \in B$ give

$$(F_\bullet, tv) \longrightarrow (F_\bullet, 0) \qquad (t \to 0).$$

Moreover, $E_\bullet \in \overline{B \cdot F_\bullet}$ by the Bruhat order on $\mathrm{FL}(V)$. Thus $x_0 \in \overline{B \cdot y}$.

Only one implication in the smoothness assertion requires proof. If $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular, its singular locus Σ is a nonempty closed B-stable subset. For any $y \in \Sigma$, the preceding observation gives $x_0 \in \overline{B \cdot y} \subseteq \Sigma$. Hence $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular at x_0. □

For $(w, \alpha) \in \mathrm{RB}(n)$, define the vector component

$$T^{\mathrm{vec}}_{w,\alpha} := T_{x_0}\overline{\mathcal{O}_{(w,\alpha)}} \cap (0 \oplus V) \subseteq T_{E_\bullet}\mathrm{FL}(V) \oplus V.$$

Lemma 3.7. *The projection π satisfies $\pi(\overline{\mathcal{O}_{(w,\alpha)}}) = X_w$, where X_w denotes the Schubert variety indexed by w, and the zero section*

$$s : X_w \longrightarrow \overline{\mathcal{O}_{(w,\alpha)}}, \qquad F_\bullet \longmapsto (F_\bullet, 0),$$

is well-defined. Consequently, there is a split exact sequence

$$0 \longrightarrow T^{\mathrm{vec}}_{w,\alpha} \longrightarrow T_{x_0}\overline{\mathcal{O}_{(w,\alpha)}} \xrightarrow{d\pi_{x_0}} T_{E_\bullet}X_w \longrightarrow 0.$$

In particular,

$$\dim T_{x_0}\overline{\mathcal{O}_{(w,\alpha)}} = \dim T_{E_\bullet}X_w + \dim T^{\mathrm{vec}}_{w,\alpha}.$$

Proof. Let $C_w = B \cdot E^w_\bullet$. Since $\pi(\mathcal{O}_{(w,\alpha)}) = C_w$, continuity gives

$$\pi(\overline{\mathcal{O}_{(w,\alpha)}}) \subseteq \overline{\pi(\mathcal{O}_{(w,\alpha)})} = X_w.$$

Conversely, scalar matrices give

$$(E^w_\bullet, tv_\alpha) \longrightarrow (E^w_\bullet, 0) \qquad (t \to 0),$$

so $(E^w_\bullet, 0) \in \overline{\mathcal{O}_{(w,\alpha)}}$. By B-stability and closedness,

$$X_w \times \{0\} = \overline{B \cdot (E^w_\bullet, 0)} \subseteq \overline{\mathcal{O}_{(w,\alpha)}}.$$

Hence $\pi(\overline{\mathcal{O}_{(w,\alpha)}}) = X_w$, and s is well-defined.

Finally, $\pi \circ s = \mathrm{id}_{X_w}$, so $d\pi_{x_0}$ is split surjective, with kernel

$$\ker\Big(d\pi_{x_0}|_{T_{x_0}\overline{\mathcal{O}_{(w,\alpha)}}}\Big) = T_{x_0}\overline{\mathcal{O}_{(w,\alpha)}} \cap (0 \oplus V) = T^{\mathrm{vec}}_{w,\alpha},$$

which proves the assertions. □

We now compute $\dim T^{\mathrm{vec}}_{w,\alpha}$, using the convention $\max(\emptyset) = 0$.

For nonempty α, let $L_\alpha = \mathbb{C}v_\alpha$ and consider the projective B-orbit

$$\mathcal{P}_{w,\alpha} := B \cdot (E^w_\bullet, L_\alpha) \subseteq \mathrm{FL}(V) \times \mathbb{P}(V).$$

We use the following theorem of Magyar.

Theorem 3.8. [Mag03, §2.2] *A pair $(F_\bullet, L) \in \mathrm{FL}(V) \times \mathbb{P}(V)$ lies in $\overline{\mathcal{P}_{w,\alpha}}$ if and only if, for all $0 \le i, j \le n$,*

$$\dim(E_i \cap F_j) \ge \dim(E_i \cap E^w_j),$$
$$\dim(E_i \cap F_j) + \dim\big(L \cap (E_i + F_j)\big) \ge \dim(E_i \cap E^w_j) + \dim\big(L_\alpha \cap (E_i + E^w_j)\big).$$

We will also use the projectivization map

$$\rho : \mathrm{FL}(V) \times (V \setminus \{0\}) \longrightarrow \mathrm{FL}(V) \times \mathbb{P}(V), \qquad (F_\bullet, v) \longmapsto (F_\bullet, [v]).$$

Since scalar matrices lie in B, we have $\mathcal{O}_{(w,\alpha)} = \rho^{-1}(\mathcal{P}_{w,\alpha})$. As ρ is an open quotient map, for $v \neq 0$ we obtain

$$(F_\bullet, v) \in \overline{\mathcal{O}_{(w,\alpha)}} \iff (F_\bullet, [v]) \in \overline{\mathcal{P}_{w,\alpha}}. \tag{3.1}$$

Lemma 3.9. *Let*

$$p := \max(\alpha), \qquad q := \max(w(\alpha)), \qquad d := \min(p, q).$$

Then

$$T^{\mathrm{vec}}_{w,\alpha} = E_d.$$

In particular, $\dim T^{\mathrm{vec}}_{w,\alpha} = d$.

Proof. If $\alpha = \emptyset$, then $v_\alpha = 0$ and

$$\mathcal{O}_{(w,\alpha)} = (B \cdot E^w_\bullet) \times \{0\}, \qquad \overline{\mathcal{O}_{(w,\alpha)}} = X_w \times \{0\}.$$

Thus $T^{\mathrm{vec}}_{w,\alpha} = 0 = E_d$. We henceforth assume that $\alpha \neq \emptyset$.

We first prove the upper bound. Since $v_\alpha \in E_q$ and E_q is B-stable,

$$\overline{\mathcal{O}_{(w,\alpha)}} \subseteq \mathrm{FL}(V) \times E_q.$$

On the other hand, $v_\alpha \in E^w_p$. The incidence variety

$$\mathcal{S}_p := \{(F_\bullet, v) \in \mathrm{FL}(V) \times V : v \in F_p\}$$

is the total space of the tautological rank-p subbundle, and hence is closed and B-stable. Therefore $\overline{\mathcal{O}_{(w,\alpha)}} \subseteq \mathcal{S}_p$. At the point $x_0 = (E_\bullet, 0)$, its pure-vector tangent space is E_p. Consequently,

$$T^{\mathrm{vec}}_{w,\alpha} \subseteq E_q \cap E_p = E_d.$$

For the reverse inclusion, we first show that

$$(E_\bullet, e_d) \in \overline{\mathcal{O}_{(w,\alpha)}}. \tag{3.2}$$

Let $L_d = \mathbb{C}e_d$. By Theorem 3.8, it is enough to verify the two rank inequalities for $(E_\bullet, L_d)$. Set

$$r_{ij} := \dim(E_i \cap E^w_j) = \#\{\ell \le j : w(\ell) \le i\}.$$

The first inequality follows from

$$r_{ij} \le \min(i, j) = \dim(E_i \cap E_j).$$

For the second, set

$$\epsilon_{ij} := \dim\big(L_\alpha \cap (E_i + E^w_j)\big),$$
$$\delta_{ij} := \dim\big(L_d \cap (E_i + E_j)\big).$$

Since these are coordinate subspaces,

$$\epsilon_{ij} = 1 \iff \text{for every } a \in \alpha, \quad w(a) \le i \text{ or } a \le j,$$

whereas $\delta_{ij} = 1$ if and only if $\max(i, j) \ge d$. Thus the required inequality is

$$\min(i, j) + \delta_{ij} \ge r_{ij} + \epsilon_{ij}.$$

It is automatic if $\delta_{ij} = 1$ or $\epsilon_{ij} = 0$. It remains to consider $\delta_{ij} = 0$ and $\epsilon_{ij} = 1$. The first condition gives

$$\max(i, j) < d = \min(p, q),$$

so $j < p$ and $i < q$. Since $p \in \alpha$, the condition $\epsilon_{ij} = 1$ and the inequality $j < p$ force $w(p) \leq i$. Likewise, choose $b \in \alpha$ with $w(b) = q$. Since $i < q = w(b)$, the same condition forces $b \leq j$.

Consequently, $i, j \geq 1$. Moreover, the value $w(p) \leq i$ occurs after position j, while the position $b \leq j$ contains the value $w(b) > i$. Hence

$$r_{ij} \leq i - 1, \qquad r_{ij} \leq j - 1,$$

and therefore $r_{ij} + 1 \leq \min(i, j)$, as required. We conclude that $(E_\bullet, L_d) \in \overline{\mathcal{P}_{w,\alpha}}$, so Equation (3.1) proves Equation (3.2).

Since $\overline{\mathcal{O}_{(w,\alpha)}}$ is B-stable and contains x_0, the curve

$$t \longmapsto (E_\bullet, te_d)$$

lies in $\overline{\mathcal{O}_{(w,\alpha)}}$ for all $t \in \mathbb{C}$. Its tangent at $t = 0$ gives $e_d \in T^{\mathrm{vec}}_{w,\alpha}$. Moreover, x_0 is fixed by B, so $T^{\mathrm{vec}}_{w,\alpha}$ is a B-stable subspace of V. For each $i < d$, an upper-triangular unipotent element sends e_d to $e_d + e_i$; hence $e_i \in T^{\mathrm{vec}}_{w,\alpha}$. It follows that $E_d \subseteq T^{\mathrm{vec}}_{w,\alpha}$, completing the proof. □

The preceding lemmas determine the tangent-space dimension at x_0.

Theorem 3.10. *For each* $(w, \alpha) \in \mathrm{RB}(n)$,

$$\dim T_{x_0} \overline{\mathcal{O}_{(w,\alpha)}} = \dim T_{E_\bullet} X_w + \min\big(\max(\alpha), \max(w(\alpha))\big).$$

Proof. Combine Lemma 3.7 with Lemma 3.9. □

3.4. **The Pattern-Avoidance Criterion for Smoothness.** We are now ready to prove the main result of this section: a complete pattern-avoidance criterion for smoothness of these B-orbit closures. Using Lemma 3.6, we reduce global smoothness to smoothness at the base point $(E_\bullet, 0)$. We then compare the dimension of the orbit closure from Lemma 3.5 with the tangent-space dimension from Theorem 3.10.

We first state the pattern avoidance criterion for smoothness for classical Schubert varieties, which gives us two of the smoothness conditions in our setting.

Theorem 3.11. [LS90] *The Schubert variety X_w is smooth if and only if w avoids both patterns* 3412 *and* 4231.

We also prove a lemma that explains the relevance of the vector contribution in the tangent-space formula:

$$\min\big(\max(\alpha), \max(w(\alpha))\big) = \dim T^{\mathrm{vec}}_{w,\alpha}.$$

Lemma 3.12. *For every $(w, \alpha) \in \mathrm{RB}(n)$, we have*

$$\min(\max(\alpha), \max(w(\alpha))) \geq |\beta|.$$

Proof. By definition, for every $i \in \beta$, there exists $j \in \alpha$ such that $i \leq_w j$. This implies

$$i \leq j \leq \max(\alpha) \quad \text{and} \quad w(i) \leq w(j) \leq \max(w(\alpha)).$$

Therefore,

$$\beta \subseteq S := \{i \in [n] \mid i \leq \max(\alpha), w(i) \leq \max(w(\alpha))\}.$$

Since w is a permutation and thus a bijection, we have $|\beta| \leq |S| \leq \max(\alpha)$ and $|\beta| \leq |S| \leq \max(w(\alpha))$. Therefore,

$$|\beta| \leq |S| \leq \min(\max(\alpha), \max(w(\alpha))).$$

□

We are now ready to prove the main theorem of this section. The proof uses purely combinatorial methods.

Theorem 3.13 (Equivalent statement of Theorem 1.6)**.** *Let $(w, \alpha) \in \mathrm{RB}(n)$. The B-orbit closure $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular if and only if (w, α) satisfies at least one of the following conditions:*

(1) the permutation w contains the pattern 3412*, regardless of color;*
(2) the permutation w contains the pattern 4231*, regardless of color;*
(3) there exists (i, j, k) where $i, k \in \alpha$ and $j \notin \alpha$ such that

$$i < j < k \quad \text{and} \quad w(k) < w(j) < w(i);$$

(4) there exists (i, j, k) where $j \in \alpha$ and $i, k \notin \alpha$ such that

$$i < j < k \quad \text{and} \quad w(k) < w(j) < w(i),$$

(4a) no $a \in \alpha$ satisfies $i < a < j$ and $w(i) < w(a)$,
(4b) no $a \in \alpha$ satisfies $k < a$ and $w(k) < w(a) < w(j)$.

Proof. By Lemma 3.6, the orbit closure $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular if and only if

$$\dim T_{x_0}\overline{\mathcal{O}_{(w,\alpha)}} > \dim \overline{\mathcal{O}_{(w,\alpha)}}.$$

Set $m := \min\big(\max(\alpha), \max(w(\alpha))\big)$. By Lemma 3.5 and Theorem 3.10, this inequality is equivalent to

$$\big(\dim T_{E_\bullet} X_w - \ell(w)\big) + \big(m - |\beta|\big) > 0.$$

The first summand is nonnegative because $\dim X_w = \ell(w)$. The second is nonnegative by Lemma 3.12. Consequently, $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular if and only if

$$\dim T_{E_\bullet} X_w - \ell(w) > 0 \quad \text{or} \quad m - |\beta| > 0.$$

The first inequality is equivalent to the singularity of X_w and, by Theorem 3.11, to condition (1) or (2).

Thus, it is sufficient to show that, given (1) and (2) are false, we have $m - |\beta| > 0$ if and only if (3) or (4) is true.

Let

$$S := \{i \in [n] \mid i \leq \max(\alpha), w(i) \leq \max(w(\alpha))\}.$$

Note that

$$\beta \subseteq S \quad \text{and} \quad |S| \leq m.$$

Therefore, we find that $m - |\beta| > 0$ is equivalent to $\beta \subsetneq S$ or $|S| < m$ being true.

First, let us consider the forward direction.

Case 1: $\beta \subsetneq S$. In this case, there must exist i such that $i \leq \max(\alpha)$, $w(i) \leq \max(w(\alpha))$, and $i \notin \beta$ (which implies $i \notin \alpha$).

Note that $\max(\alpha) > i$ since $i \notin \alpha$ tells us $i \neq \max(\alpha)$. This forces $w(\max(\alpha)) < w(i)$ since $i \notin \beta$. We also know that $w(i) < \max(w(\alpha))$ since $i \notin \alpha$ tells us $w(i) \neq \max(w(\alpha))$. This forces $w^{-1}(\max(w(\alpha))) < i$ since $i \notin \beta$. Therefore, we have

$$w^{-1}(\max(w(\alpha))) < i < \max(\alpha) \quad \text{and} \quad w(\max(\alpha)) < w(i) < \max(w(\alpha)).$$

Note that $i \notin \alpha$ but $w^{-1}(\max(w(\alpha))), \max(\alpha) \in \alpha$. Therefore, condition (3) is satisfied.

Case 2: $|S| < m$. Let

$$A := \{\, i \in [n] \mid i \leq \max(\alpha),\ w(i) > \max(w(\alpha)) \,\},$$
$$B := \{\, i \in [n] \mid i > \max(\alpha),\ w(i) \leq \max(w(\alpha)) \,\}.$$

Note that $|A| = \max(\alpha) - |S|$ and $|B| = \max(w(\alpha)) - |S|$. Since $|S| < m$, we know both A and B are nonempty. Thus, we can let $a := \min(A)$ and $b := w^{-1}(\min(w(B)))$. Since $a \in A$, we have $w(a) > \max(w(\alpha))$, which tells us that $a \notin \alpha$. Similarly, since $b \in B$, we have $b > \max(\alpha)$, which tells us that $b \notin \alpha$.

Note that $\min(\alpha) \neq a$ since $a \notin \alpha$. So, we can take the following two subcases.

Subcase 1: $\min(\alpha) > a$. So, we know

$$a < \min(\alpha) < b.$$

From the definition of A and B, we have $w(b) \leq \max(w(\alpha)) < w(a)$. Since $b \notin \alpha$, we have $w(b) \neq \max(w(\alpha))$. Since α is an antichain, we have $\max(w(\alpha)) = w(\min(\alpha))$. So, we get

$$w(b) < w(\min(\alpha)) < w(a).$$

For all $c \in \alpha$, we have $w(c) \leq \max(w(\alpha)) = w(\min(\alpha))$ so $(a, \min(\alpha), b)$ satisfies (4a). Additionally, we have $c \leq \max(\alpha) < b$, so $(a, \min(\alpha), b)$ satisfies (4b). Therefore, the triple $(a, \min(\alpha), b)$ satisfies (4).

Subcase 2: $\min(\alpha) < a$. Then, by the definition of A and B, we have

$$\min(\alpha) < a < \max(\alpha) < b.$$

Assume for the sake of contradiction that $w(\max(\alpha)) < w(b)$. We have $\max(w(\alpha)) < w(a)$ since $a \in A$. We also know $w(\min(\alpha)) = \max(w(\alpha))$ since α is an antichain. Combining $w(b) \neq \max(w(\alpha))$ since $b \notin \alpha$ with $b \in B$, we know $w(b) < \max(w(\alpha))$. Therefore, we get

$$w(\max(\alpha)) < w(b) < w(\min(\alpha)) < w(a).$$

This contradicts (1), so $w(b) \leq w(\max(\alpha))$; since $b \neq \max(\alpha)$, we have $w(b) < w(\max(\alpha))$.

Note that $\max(w(\alpha)) < w(a)$ since $a \in A$. So,

$$a < \max(\alpha) < b \quad \text{and} \quad w(b) < w(\max(\alpha)) \leq \max(w(\alpha)) < w(a).$$

Additionally, for all $c \in \alpha$, we have $w(c) \leq \max(w(\alpha)) < w(a)$ so $(a, \max(\alpha), b)$ satisfies (4a) and $c \leq \max(\alpha) < b$ so $(a, \max(\alpha), b)$ satisfies (4b). Therefore, the triple $(a, \max(\alpha), b)$ satisfies (4).

Now, let us consider the backward direction.

Case 1: Let (i, j, k) satisfy (3).

Then, we have $j < k \leq \max(\alpha)$ and $w(j) < w(i) \leq \max(w(\alpha))$, so $j \in S$. Assume for the sake of contradiction that $j \in \beta$, implying there exists $a \in \alpha$ such that $j <_w a$. Consider the following subcases on a.

Subcase 1: $j < a < k$. Since $i, a, k \in \alpha$, then $w(k) < w(a) < w(i)$. We also know that $w(j) < w(a)$ since $j <_w a$ and $w(k) < w(j)$ since (i, j, k) satisfies (3). So, we get

$$i < j < a < k \quad \text{and} \quad w(k) < w(j) < w(a) < w(i).$$

However, this contradicts (2) being false.

Subcase 2: $a = k$. Then, we have $w(j) < w(k)$ since $j <_w a$, contradicting (i, j, k) satisfying (3).

Subcase 3: $k < a$. Since $a, k \in \alpha$, we require $w(a) < w(k)$ since α is an antichain. However, we know that $w(j) < w(a)$ since $j <_w a$ and $w(k) < w(j)$ since (i, j, k) satisfies (3). This tells us $w(k) < w(j) < w(a)$, contradicting $w(a) < w(k)$.

Therefore, we have $j \notin \beta$ but $j \in S$, implying $\beta \subsetneq S$ and thus $m - |\beta| > 0$.

Case 2: Let (i, j, k) satisfy (4).

Assume for the sake of contradiction that $i \in \beta$, which implies that there exists $a \in \alpha$ such that $i \leq a$ and $w(i) \leq w(a)$. Since $i \notin \alpha$, this forces $i < a$ and $w(i) < w(a)$. Therefore, we get $w(j) < w(i) < w(a)$. Since $a, j \in \alpha$, this implies $a < j$. So, we have

$$i < a < j \quad \text{and} \quad w(i) < w(a).$$

This contradicts (4a); hence, we have $i \notin \beta$. However, since $i < j \leq \max(\alpha)$, we know $i \in [\max(\alpha)]$. Therefore, we get $\beta \neq [\max(\alpha)]$. But since $\beta \subseteq [\max(\alpha)]$, we have $|\beta| \neq \max(\alpha)$.

Similarly, assume for the sake of contradiction that $k \in \beta$, so there exists $b \in \alpha$ such that $k \leq b$ and $w(k) \leq w(b)$. Since $k \notin \alpha$, this forces $k < b$ and $w(k) < w(b)$. Therefore, we get $j < k < b$. Since $j, b \in \alpha$, this implies $w(b) < w(j)$. So, we have

$$k < b \quad \text{and} \quad w(k) < w(b) < w(j).$$

This contradicts (4b); hence, we have $k \notin \beta$, so $w(k) \notin w(\beta)$. However, since $w(k) < w(j) \leq \max(w(\alpha))$, we know $w(k) \in [\max(w(\alpha))]$. Therefore, we get $w(\beta) \neq [\max(w(\alpha))]$. But since $w(\beta) \subseteq [\max(w(\alpha))]$, we have $|\beta| \neq \max(w(\alpha))$.

Since $|\beta| \neq \max(\alpha)$ and $|\beta| \neq \max(w(\alpha))$, we get $|\beta| \neq m$. We already know $m - |\beta| \geq 0$ by Lemma 3.12, so $|\beta| \neq m$ tells us $m - |\beta| > 0$.

Therefore, we are done. □

Theorem 1.6 is an equivalent statement of Theorem 3.13 in terms of colored permutation pattern avoidance.

4. Duality and Future Directions

The type II theta correspondence model and the basic Fourier–Jacobi model are expected to be an instance of relative Langlands duality. A central question for any such dual pair is how geometric invariants of one side transfer to the other, and smoothness of orbit closures is a natural invariant to test.

Using the two pattern avoidance criteria we have obtained, we find that smoothness is not preserved in general under the bijection between the spherical varieties. However, we can completely characterize the Fourier–Jacobi orbit closures that map to singular matrix Schubert varieties in terms of the containment of four colored patterns. As a corollary, we obtain a full characterization of failures for one direction of the smoothness duality: if a Fourier–Jacobi orbit closure is singular for any reason other than a 4231 pattern colored as 4231, then the corresponding matrix Schubert variety is singular as well. This, however, is not the full picture. We discuss the remaining open questions at the end of this section.

The geometric duality between Fourier-Jacobi orbits and theta orbits is given by [FGT25, Lemma 4.25] and [FGT25, Lemma 4.27]. Finkelberg, Ginzburg, and Travkin construct the bijection through an intermediate combinatorial object, known as a maze. However, specializing to the square, general linear case we are working with, the maze machinery degenerates simply.

Definition 4.1. Define a natural bijection

$$\Psi : \mathrm{PM}(n,n) \to \mathrm{RB}(n)$$

which turns a partial permutation σ into an antichain by pairing the empty columns with the empty rows of the matrix representation decreasingly, i.e., the smallest indexed empty column in the partial permutation matrix is paired with the largest indexed empty row until the matrix is full rank. The column indices of the added 1's are exactly α, and the resulting permutation matrix defines a permutation w. Then,

$$\Psi(\sigma) = (w, \alpha).$$

The inverse map is defined by writing w as a permutation matrix and then removing all 1's with column indexed by α (i.e., the red 1's) to get the matrix representation of a partial permutation $\sigma \in \mathrm{PM}(n,n)$.

Note that the definition of Ψ forces the constructed (w, α) to satisfy the antichain property: the added 1's are decreasing in row index.

Example 4.2. Consider $[(2,1),(3,3)] \in \mathrm{PM}(4,4)$, which is denoted in our column-first convention for partial matchings in one-line notation. Then,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi([(2,1),(3,3)]) = 4132.$$

We now utilize our smoothness criterion for matrix Schubert varieties to prove a pattern avoidance criterion for the corresponding Fourier–Jacobi orbit closures.

Lemma 4.3. *Let $\sigma \in \mathrm{PM}(n,n)$ and $\Psi(\sigma) = (w, \alpha) \in \mathrm{RB}(n)$. Then $\overline{\mathcal{O}_\sigma}$ is singular if and only if (w, α) contains at least one of the following patterns:*

$$21,\ 231,\ 231,\ 312.$$

Proof. Recall the smoothness criterion for matrix Schubert varieties $\overline{\mathcal{O}_\sigma}$ from Theorem 1.2. The nonzero 2×2 condition is equivalent to the existence of an entry (r, c) in M_σ such that

(1) $(r, j) = 0$ for all $1 \le j \le c$,
(2) $(i, c) = 0$ for all $r \le i \le n$,
(3) there exists some $(r', c') = 1$ for $r' \ge r + 1$ and $c' \le c - 1$.

The entry (r, c) in M_σ has the same position as the top right entry of the nonzero 2×2 block in $\mathrm{rk}(M_\sigma)$.

We first prove the reverse direction. Recall the following two properties of the colored permutation matrix of (w, α):

- under Ψ^{-1}, the red 1's are deleted to obtain M_σ,
- each row/column has exactly one 1, whether that be red or blue; in particular, if a row/column already has a 1 somewhere, then the rest of that row/column is empty.

The four diagrams in Figure 2 show how the four patterns from the lemma statement force a nonzero 2×2 in the southwest rank table of $\Psi^{-1}((w, \alpha)) = M_\sigma$. The black dotted outline shows where this 2×2 is and the dotted green lines represent empty entries that satisfy (1) and (2). Notice that each figure has a blue 1 to the southwest of the green lines, which satisfies (3).

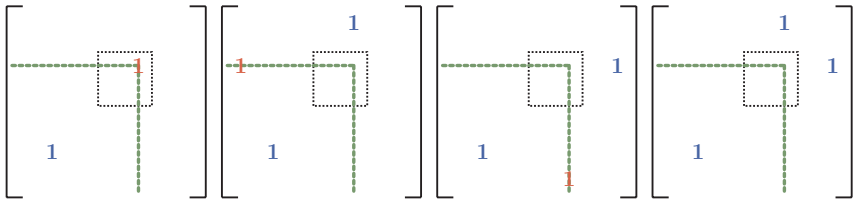


FIGURE 2. The four patterns 21, 231, 231, 312 (left to right) each force a nonzero 2×2 in the southwest rank table of $\Psi^{-1}((w, \alpha)) = M_\sigma$.

More specifically, for each pattern, label the 1's left to right as (r_i, c_i). Table 2 gives the entry satisfying the three conditions to form a nonzero 2×2.

TABLE 2. Where the four colored patterns create a 2×2 in the dual matrix Schubert variety.

Pattern	21	231	231	312
Violating Entry	(r_2, c_2)	(r_1, c_3)	(r_3, c_2)	(r_3, c_2)

For the forward direction, if $\overline{\mathcal{O}_\sigma}$ is singular, then there exists some (r, c) satisfying all three conditions. We do casework on the color and position of the 1's in row r and column c and show that all cases force $\Psi(\sigma)$ to have one of the patterns

$$21,\ 231,\ 231,\ 312.$$

We give labels to five regions in Figure 3, outlined in orange. The 2×2 in M_σ still remains outlined in black.

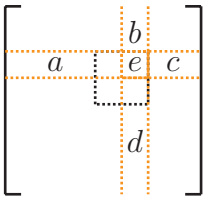


FIGURE 3. The 5 regions where 1's in row r and column c may reside.

Note that a colored permutation matrix of $\Psi(\sigma) = (w, \alpha) \in \mathrm{RB}(n)$ has exactly one 1 (of either color) among the regions a, e, and c. Similarly, a colored permutation matrix has exactly one 1 (of either color) among the regions b, e, and d. Furthermore, by (1) and (2), there can be no blue 1's in regions a, e, and d.

The existence of a blue 1 strictly southwest of e is guaranteed by (3). Then, if there is a red 1 in region b or region c, then $\Psi(\sigma)$ contains a 21 pattern.

We now do casework for all remaining combinations of blue 1's in regions b and c and red 1's in regions a, e, and d.

Case 1: blue 1 in b and blue 1 in c. This forces a 312 in $\Psi(\sigma)$.

Case 2: red 1 in region a and blue 1 in region b. This splits into two subcases according to whether the red 1 in region a is to the left or right of the guaranteed blue 1 strictly southwest of e. The former forces a 231, while the latter forces a 21.

Case 3: red 1 in region d and blue 1 in region c. This splits into two subcases according to whether the red 1 in region d is above or below the guaranteed blue 1 strictly southwest of e. The former forces a 21, while the latter forces a 231.

Case 4: red 1 in region a and red 1 in region d. This case is impossible, as such a positioning contradicts that α is an antichain. The red 1's must have decreasing row indices from left to right.

Case 5: red 1 in region e. This forces a 21.

All these cases are collected in Figure 4. □

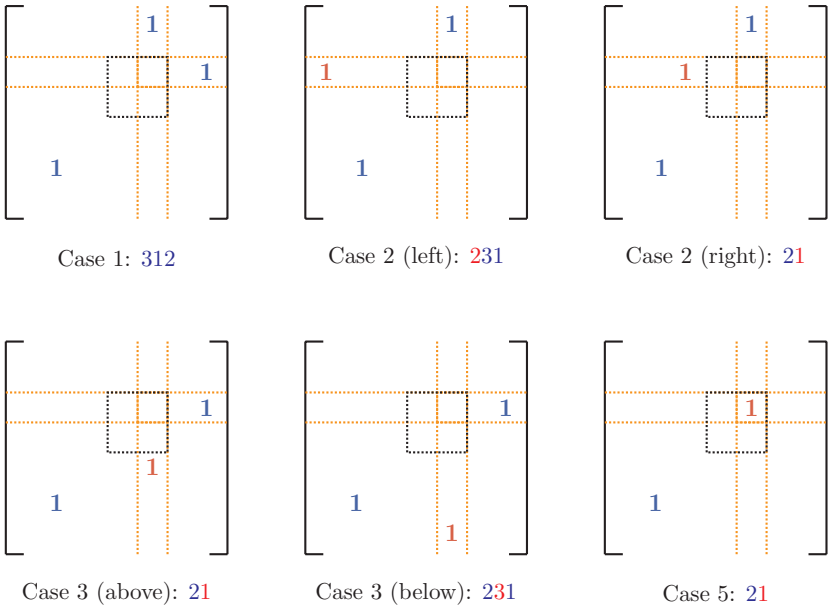


FIGURE 4. Forward-direction casework across placements of colored 1's within the five regions a, b, c, d, e of Figure 3. The colored pattern in each case caption shows which one of the four colored patterns is forced.

As a corollary, we obtain a full characterization of failures for one direction of the proposed smoothness duality.

Corollary 4.4. Let $(w, \alpha) \in \mathrm{RB}(n)$ and $\sigma = \Psi^{-1}((w, \alpha)) \in \mathrm{PM}(n, n)$. If $\overline{\mathcal{O}_{(w,\alpha)}}$ is singular for any reason other than a 4231 pattern colored as 4231, then $\overline{\mathcal{O}_\sigma}$ is singular as well.

Proof. Using Lemma 4.3, we go one by one through the singularity conditions of $\overline{\mathcal{O}_{(w,\alpha)}}$ from Theorem 1.6, showing that all of them other than 4231 force $\overline{\mathcal{O}_\sigma}$ to be singular by containing one of the patterns

$$21,\ 231,\ 231,\ 312.$$

(1) Table 3 shows all valid colorings of the 3412 pattern along with the singular pattern they contain.

TABLE 3. All colorings of 3412 and their corresponding pattern that creates a 2×2 in the dual matrix Schubert variety.

Coloring	3412	3412	3412	3412	3412	3412	3412	3412	3412
Pattern	312	231	231	21	21	21	21	21	21

(2) Table 4 shows all valid colorings of the 4231 pattern. All colorings except 4231 contain a pattern that makes $\overline{\mathcal{O}_\sigma}$ singular.

TABLE 4. All colorings of 4231 and their corresponding pattern that creates a 2×2 in the dual matrix Schubert variety.

Coloring	4231	4231	4231	4231	4231	4231	4231	4231	4231	4231	4231	4231
Pattern	312	none	21	21	21	231	231	21	21	21	21	21

(3) This condition is equivalent to the pattern 321, which contains 21.
(4) This condition is equivalent to the pattern 321 along with some extra conditions, but the pattern 321 already contains 21.

□

While this gives a complete combinatorial account of one direction of the correspondence, the full picture remains unclear. For example, Corollary 4.4 controls only one implication: singularity on the Fourier–Jacobi side forces singularity on the theta side, up to a single exceptional pattern. The reverse implication fails much more drastically: by Theorem 1.6 and Lemma 4.3, singular matrix Schubert varieties are abundant while their duals are frequently smooth. We do not know whether these failures admit any comparably clean description. We also do not yet have a conceptual explanation for the exceptional role played by the colored pattern 4231. A natural direction for further investigation would involve comparing the Kazhdan–Lusztig polynomials of the two spherical varieties [KL79]. We therefore regard the results of this section as preliminary evidence about how smoothness interacts with relative Langlands duality.

Another direction for future work regards counting the smooth basic Fourier–Jacobi orbit closures. Having characterized which orbit closures $\overline{\mathcal{O}_{(w,\alpha)}}$ are smooth in Section 3, it is natural to count them for a fixed dimension, as we did for matrix Schubert varieties in Section 2.4. Here, the enumeration appears more complex, being related to the number of smooth Schubert varieties of a fixed dimension.

Conjecture 4.5. *For each n, the number of colored permutations $(w, \alpha) \in \mathrm{RB}(n)$ whose orbit closure $\overline{\mathcal{O}_{(w,\alpha)}}$ is smooth equals*

$$(2n+2)\,L_n - \sum_{i=0}^{n} L_i\,L_{n-i},$$

where L_n denotes the number of smooth Schubert varieties X_w for $w \in S_n$; that is, the number of permutations in S_n avoiding 3412 *and* 4231.

An interesting observation is that the term $\sum_{i=0}^{n} L_i\,L_{n-i}$ has the form of the Catalan recursion, which was used to count smooth matrix Schubert varieties. Here, the value L_n has no closed form, but the generating function $V(t)$ for the sequence $\{L_n\}$ has the following closed form [BMB07]:

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3}.$$

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